ECE 203 - Midterm Solutions - Fall 2022

1 Question 1

Let A= {The student likes math.}, and G_i = {group i is chosen.} for $i = 1, 2, 3$. Then by the law of total probability, we have

$$\begin{aligned} P[A] &= P[A|G_1] \times P[G_1] + P[A|G_2] \times P[G_2] + P[A|G_3] \times P[G_3] \\ &= \frac{5}{20} \times \frac{20}{20+12+18} + \frac{7}{12} \times \frac{12}{20+12+18} + \frac{3}{18} \times \frac{18}{20+12+18} \\ &= \frac{5}{20+12+18} + \frac{7}{20+12+18} + \frac{3}{20+12+18} \\ &= \frac{15}{50} \\ &= 0.3. \end{aligned}$$

2 Question 2

Define the events below.

K = {a total of k balls are withdrawn with r red balls.}

$K1$ = {a total of $k-1$ balls are withdrawn with $r-1$ red balls.}

R = {the k^{th} withdrawn ball is red.}

If $k < r$ or $k > n+m$, then $p[K] = 0$.

Assume $r \leq k \leq n+m$ and we draw $k-1$ balls sequentially. The probability of drawing $r-1$ red balls and $k-r$ blue balls is given by

$$\begin{aligned} P[K1] = \binom{k-1}{r-1} &\times \frac{n}{n+m} \times \frac{n-1}{n+m-1} \times ... \times \frac{n-r+2}{n+m-r+2} \times \frac{m}{n+m-r+1} \\ &\times \frac{m-1}{n+m-r} \times ... \times \frac{m-k+r+1}{n+m-k+2}, \end{aligned}$$

and the probability of choosing a red ball after we have drawn $k-1$ balls with $r-1$ red balls is

$$P[R] = \frac{n-r+1}{n+m-k+1}.$$

Thus, we have

$$
\begin{aligned}
P[K] &= P[K1] \times P[R] \\
&= \binom{k-1}{r-1} \times \frac{\Big(n(n-1)...(n-r+1)\Big)\Big(m(m-1)...(m-k+r+1)\Big)}{(n+m)(n+m-1)...(n+m-k+1)} \\
&= \binom{k-1}{r-1} \times \frac{\Big(n!/(n-r)!\Big)\Big(m!/(m-(k-r))!\Big)}{(n+m)!/(n+m-k)!} \\
&= \binom{k-1}{r-1} \times \frac{\Big(r!\binom{n}{r}\Big)\Big((k-r)!\binom{m}{k-r}\Big)}{k!\binom{n+m}{k}} \\
&= \frac{r}{k} \times \frac{\binom{n}{r}\binom{m}{k-r}}{\binom{n+m}{k}}.
\end{aligned}
$$

3 Question 3

By the law of total probability, we have

$$
\begin{aligned}
P[Black] &= P[Black|Box1] \times P[Box1] + P[Black|Box2] \times P[Box2] \\
&= \frac{1}{2} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2} \\
&= \frac{7}{12}.
\end{aligned}
$$

By Bayes' law, we have

$$
\begin{aligned}
P[Box1|White] &= \frac{P[White|Box1] \times P[Box1]}{P[White]} \\
&= \frac{P[White|Box1] \times P[Box1]}{1 - P[Black]} \\
&= \frac{1/2 \times 1/2}{1 - 7/12} \\
&= \frac{1/4}{5/12} \\
&= \frac{3}{5}.
\end{aligned}
$$