Solutions

Question 1: Law of Total Probability

Given:

$$P(A) = 0.3, \quad P(B) = 0.5, \quad P(C) = 0.2$$
$$P(D|A) = 0.05, \quad P(D|B) = 0.02, \quad P(D|C) = 0.01$$

(a) Find $P(D)$:

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ &= 0.05 \times 0.3 + 0.02 \times 0.5 + 0.01 \times 0.2 \\ &= 0.015 + 0.010 + 0.002 = \boxed{0.027} \end{aligned}$$

(b) Find $P(A|D)$:

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{0.05 \times 0.3}{0.027} = \frac{0.015}{0.027} = \boxed{\frac{5}{9} \approx 0.5556}$$

Question 2: Multiplication Rule

Given:

$$P(A) = 0.6, \quad P(B|A) = 0.7, \quad P(C|A \cap B) = 0.8$$

(a) Compute $P(A \cap B \cap C)$:

$$\begin{aligned} P(A \cap B \cap C) &= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \\ &= 0.6 \times 0.7 \times 0.8 = \boxed{0.336} \end{aligned}$$

(b) $P(C|A \cap B)$:

$$\boxed{0.8} \quad \text{(given directly)}$$

Question 3: Bayes' Theorem (ECE Context)

Given:

$$P(H_0) = 0.5, \quad P(H_1) = 0.5$$
$$P(R_0|H_0) = 0.9, \quad P(R_0|H_1) = 0.2$$

(a) Compute $P(H_0|R_0)$:
First, find $P(R_0)$:

$$\begin{aligned} P(R_0) &= P(R_0|H_0)P(H_0) + P(R_0|H_1)P(H_1) \\ &= 0.9 \times 0.5 + 0.2 \times 0.5 = 0.45 + 0.10 = 0.55 \end{aligned}$$

Now apply Bayes' theorem:

$$P(H_0|R_0) = \frac{P(R_0|H_0)P(H_0)}{P(R_0)} = \frac{0.9 \times 0.5}{0.55} = \frac{0.45}{0.55} = \boxed{\frac{9}{11} \approx 0.818}$$

(b) Interpretation: The probability is about 81.8%, so receiving a 0 strongly suggests that 0 was sent.

Bonus Question: Mean Value

Given PMF:

$$P(X = 0) = 0.1, \quad P(X = 1) = 0.3, \quad P(X = 2) = 0.4, \quad P(X = 3) = 0.2$$

(a) Validity:

$$0.1 + 0.3 + 0.4 + 0.2 = \boxed{1.0}$$

Hence, this is a valid PMF.

(b) Compute $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \sum_x x\, P(X = x) = 0 \cdot 0.1 + 1 \cdot 0.3 + 2 \cdot 0.4 + 3 \cdot 0.2$$

$$= 0 + 0.3 + 0.8 + 0.6 = \boxed{1.7}$$

(c) Interpretation: On average, the sensor detects 1.7 pulses per interval, which is the long-run average measurement.