ECE 203
Probability Theory and Statistics I
Tutorial 3

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Problem 1

Alice and Bob have a pair of biased coins:

$$P[hh] = 0.2 \qquad P[ht] = 0.3$$
$$P[th] = 0.3 \qquad P[tt] = 0.2$$

Let

$A = \{\text{first coin is heads}\}$

$B = \{\text{second coin is heads}\}$

a) Are the events $\{hh\}, \{ht\}, \{th\}, \{tt\}$ independent?

b) Are the events A and B independent?

c) Can you find two events that are independent (other than the trivial case of $\emptyset$ and S)?

Problem 1 - Solution

a) No, they are not. If I know hh has occured, I know that ht has not occured. So hh and ht are not independent. So the 4 events are not independent. Specifically:

$$P[hh]P[ht] = 0.2 \times 0.3 \neq 0 = P[\{hh\} \cap \{ht\}]$$

b) No.

$$P[A] = 0.2 + 0.3$$
$$P[B] = 0.2 + 0.3$$

but $P[A]P[B] = 0.5 \times 0.5 \neq 0.2 = P[AB] = P[hh]$.

Problem 1 - Solution

c) Let $C = \{ht, hh\}$ and $D = \{hh, tt\}$. Then

$$P[C]P[D] = 0.5 \times 0.4 = 0.2 = P[hh] = P[CD]$$

Also, note that

$$P[C|D] = P[CD]/P[D] = 0.2/0.4 = 0.5 = P[C]$$
$$P[D|C] = P[CD]/P[C] = 0.2/0.5 = 0.4 = P[D]$$

Problem 2

- In a factory, units are manufactured by machines H_1,H_2,H_3 in the proportions 25, 35, and 40. The percentages are 5%, 4%, and 2%, respectively, of the manufactured units that are defective. The units are mixed and sent to the customers.

a) Find the probability that a randomly chosen unit is defective.

b) Suppose that a customer discovers that a certain unit is defective. What is the probability that it has been manufactured by machine H_1?

Problem 2 - Solution

- In a factory, units are manufactured by machines H_1,H_2,H_3 in the proportions 25, 35, and 40. The percentages are 5%, 4%, and 2%, respectively, of the manufactured units that are defective. The units are mixed and sent to the customers. (a) Find the probability that a randomly chosen unit is defective. (b) Suppose that a customer discovers that a certain unit is defective. What is the probability that it has been manufactured by machine H_1?

- a) The law of total probability:

$$H_i = \text{“unit produced by machine } H_i\text{”} \qquad \text{A=“unit is defective”}$$

$$P(A) = \sum_{i=1}^{n} P(H_i)P(A|H_i) \qquad P(A) = 0.25 \cdot 0.05 + 0.35 \cdot 0.04 + 0.40 \cdot 0.02 = 0.0345.$$

- b) Bayes' Theorem

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{\sum_{j=1}^{n} P(H_j)P(A|H_j)} \qquad P(H_1|A) = \frac{0.25 \cdot 0.05}{0.25 \cdot 0.05 + 0.35 \cdot 0.04 + 0.40 \cdot 0.02} = 0.36$$

Problem 3

- Suppose A and B are independent events. Does it follow that A^c and B^c are also independent? That is, does $P(AB) = P(A)P(B)$ guarantee that $P(A^cB^c) = P(A^c)P(B^c)$?

Problem 3 - Solution

- Suppose A and B are independent events. Does it follow that A^c and B^c are also independent? That is, does $P(AB) = P(A)P(B)$ guarantee that $P(A^cB^c) = P(A^c)P(B^c)$?
- The answer is yes, the proof being accomplished by equating two different expressions for $P(A^cB^c)$. First, we know that

$$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^cB^c)$$

But the union of two complement is also the complement of their intersection. Therefore,

$$P(A^c \cup B^c) = 1 - P(AB)$$

Combining the two equations above, we get

$$1 - P(AB) = 1 - P(A) + 1 - P(B) - P(A^cB^c)$$

Since A and B are independent, $P(AB) = P(A) \cdot P(B)$, so

$$\begin{aligned} P(A^cB^c) &= 1 - P(A) + 1 - P(B) - (1 - P(A)P(B)) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c) \end{aligned}$$

Problem 4

Suppose you have a hypothesis H. So either H or H^c occurs,

$$H = \{\text{there is a plane in the sky}\}$$

You now observe event E_1 occurs and know prior $P[H]$ and both $P[E_1|H]$ and $P[E_1|H^c]$.

a) What is posterior $P[H|E_1]$ in terms of what you know?

b) Now suppose you observe that E_2 also occured. What is posterior $P[H|E_1 E_2]$?

c) Suppose that E_1 and E_2 are i) conditionally independent given H, and ii) conditionally independent given H^c. Express the posterior $P[H|E_1 E_2]$ in terms of $P[H|E_1]$.

Problem 4 - Solution

- a) Deriving Baye's rule from first principles:

$$
\begin{aligned}
P[H|E_1] &= \frac{P[HE_1]}{P[E_1]} \\
&= \frac{P[E_1|H]P[H]}{P[E_1|H]P[H] + P[E_1|H^c]P[H^c]}
\end{aligned}
$$

b) This is the same as a), except we replace E_1 with E_2E_1:

$$
P[H|E_2E_1] = \frac{P[E_2E_1|H]P[H]}{P[E_2E_1|H]P[H] + P[E_2E_1|H^c]P[H^c]}
$$

Problem 4 - Solution

$$P[E_2E_1|H] = P[E_2|H]P[E_1|H]$$
$$P[E_2E_1|H^c] = P[E_2|H^c]P[E_1|H^c]$$

So

$$\begin{aligned}
P[H|E_2E_1] &= \frac{P[E_2E_1|H]P[H]}{P[E_2E_1|H]P[H] + P[E_2E_1|H^c]P[H^c]} \\
&= \frac{P[E_2|H]P[E_1|H]P[H]}{P[E_2|H]P[E_1|H]P[H] + P[E_2|H^c]P[E_1|H^c]P[H^c]} \\
&= \frac{P[E_2|H]\frac{P[E_1|H]P[H]}{P[E_1]}}{P[E_2|H]\frac{P[E_1|H]P[H]}{P[E_1]} + P[E_2|H^c]\frac{P[E_1|H^c]P[H^c]}{P[E_1]}} \\
&= \frac{P[E_2|H]P[H|E_1]}{P[E_2|H]P[H|E_1] + P[E_2|H^c]P[H^c|E_1]}
\end{aligned}$$

Problem 4 - Solution

Basically, same as Baye's rule, but:

- $P[E_1|H]$ replaced with $P[E_2|H]$ (since we want to incorporate effect of event E_2)
- $P[H]$ replaced with $P[H|E_1]$

This technique lets you update in an 'online' manner the probability of H as observations $E_1, E_2, E_3, \ldots$ are collected. You just update your last posterior probability $P[H|E_1 \cdots E_{n-1}]$ by applying Baye's rule with $P[H|E_1 \cdots E_{n-1}]$ as your prior and $P[E_n|H]$ and $P[E_n|H^c]$.