ECE 203
Probability Theory and Statistics I
Tutorial 4

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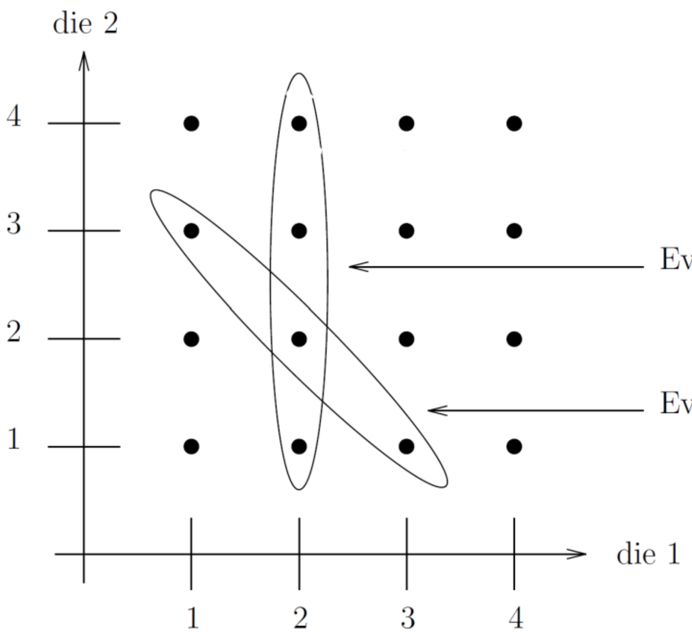
UNIVERSITY OF WATERLOO | FACULTY OF ENGINEERING

Problem 1

We throw a fair four-sided die twice. Let E be the event that "the sum of the dice is 4", and let F be the event that "the first die thrown has value 2". Are the events E and F independent?

Problem 1 - Solution



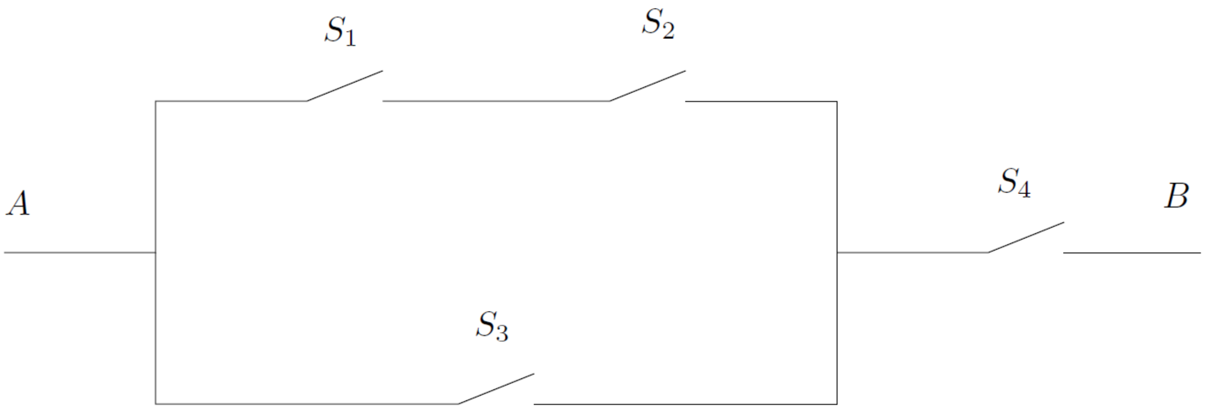
$p(E) = 3/16$ and $P(F) = 4/16$, thus

$$\frac{3}{16} \cdot \frac{4}{16} = \frac{3}{64} \neq \frac{1}{16}$$

The events E and F are therefore not independent.

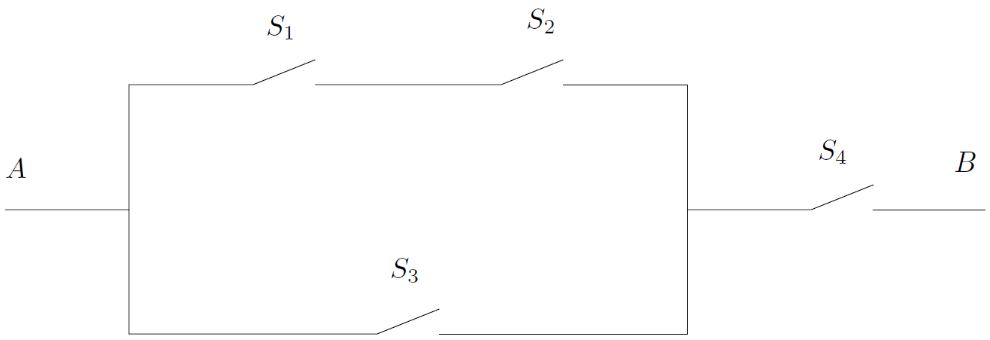
Problem 2

Consider the following connection of switches:



Define the event: $E_i, i = 1, 2, 3, 4$ as: Switch S_i is closed. Assume that $P(E_1) = P(E_2) = P(E_3) = P(E_4) = a$. Define the event $\mathcal{E}$ as the event that point A is connected to point B. Compute the probability of the event $\mathcal{E}$.

Problem 2 - Solution



E_1: Switch S_1 is closed. $\implies$ E_1^c: Switch S_1 is open.

E_2: Switch S_2 is closed. $\implies$ E_2^c: Switch S_2 is open.

E_3: Switch S_3 is closed. $\implies$ E_3^c: Switch S_3 is open.

E_4: Switch S_4 is closed. $\implies$ E_4^c: Switch S_4 is open.

Assume that $P(E_1) = P(E_2) = P(E_3) = P(E_4) = a$, and consequently, $P(E_1^c) = P(E_2^c) = P(E_3^c) = P(E_4^c) = 1 - a$.

Note that switch closures are independent of each other.

$$\mathcal{E} = [(E_1E_2) \cup E_3]E_4$$

$$P(\mathcal{E}) = P[(E_1E_2) \cup E_3] \times P(E_4)$$

$$P[(E_1E_2) \cup E_3] = P(E_1E_2) + P(E_3) - P(E_1E_2E_3) = a + a^2 - a^3$$

$$P(\mathcal{E}) = a(a + a^2 - a^3) = a^2 + a^3 - a^4$$

Problem 3

The number of customers who enter a small shop at a given hour is a discrete random variable X with the following probability mass function:

x (customer/hour)	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

a) Verify that this defines a valid PMF.

b) Compute the expected number of customers per hour, E[X].

c) Interpret the result briefly in context.

Problem 3 - Solution

a) All probabilities are not negative. Also, we should have $\sum_{x \in \mathcal{X}} p_X(x) = 1$.

$0.1 + 0.2 + 0.4 + 0.2 + 0.1 = 1 \longrightarrow$ valid PMF

b)

$$\mathbb{E}[X] = \sum_x x \cdot P(X = x) = 0(0.1) + 1(0.2) + 2(0.4) + 3(0.2) + 4(0.1)$$

$$= 0 + 0.2 + 0.8 + 0.6 + 0.4 = 2.0$$

c) On average, the shop receives **2 customers per hour**.

Problem 4

The discrete random variable U has a geometric distribution of the form

$$P(U = j) = ap^j \ , \quad j = 0, 1, 2, \cdots$$

If $P(U \geq 4) = 1/256$, find $P(U \geq 2)$.

Problem 4 - Solution

Since the given distribution must sum to unity we have $\sum_{j=0}^{\infty} ap^j = 1 = a/(1-p)$ which implies that $a = 1 - p$. Now

$$P(U \geq 4) = \sum_{j=4}^{\infty} ap^j = 1/256 = \frac{ap^4}{1-p} = \frac{1}{256} \quad \Rightarrow \quad p = \frac{1}{4}$$

Then $P(U \geq 2) = (1-p)p^2/(1-p) = 1/16$.

Problem 5

Let the PMF of X be given by

$$p_X(x) = ax^2 \qquad \text{for } x = -2, -1, 0, 1, 2, 3$$

a) Find a.

b) What is $P[-1 < X \leq 2]$?

c) What is $E[X]$?

Problem 5 - Solution

a)

$$
\begin{aligned}
p_X(-2) &= 4a \\
p_X(-1) &= a \\
p_X(0) &= 0 \\
p_X(1) &= a \\
p_X(2) &= 4a \\
p_X(3) &= 9a.
\end{aligned}
$$

These must sum to 1. So $4a + a + 0 + a + 4a + 9a = 19a = 1 \rightarrow a = 1/19$.

b) $P[-1 < X \leq 2] = P[X \in \{0, 1, 2\}] = P_X(0) + P_X(1) + P_X(2) = 5/19$.

c)

$$
\begin{aligned}
E[X] &= \sum_x p_X(x)x \\
&= 4a \times -2 + a \times -1 + 0 \times 0 + a \times 1 + 4a \times 2 + 9a \times 3 \\
&= 27/19
\end{aligned}
$$