ECE 203
Probability Theory and Statistics I
Tutorial 4

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Problem 1

The nth central moment of a r.v. X is defined by

$$\mu_{X,n} = E[\ (X - E[X])^n\ ]$$

a) Find $\mu_{X,0}$, $\mu_{X,1}$ and $\mu_{X,2}$.

b) Let $Y = aX + b$. What is the n-th central moment $\mu_{Y,n}$ of Y?

Problem 1 - Solution

a)

$$\mu_{X,0} = E[\ (X - E[X])^0\ ] = E[1] = 1$$
$$\mu_{X,1} = E[\ (X - E[X])^1\ ] = E[X] - E[X] = 0$$
$$\mu_{X,2} = E[\ (X - E[X])^2\ ] = Var[X]$$

b)

$$\begin{aligned}
\mu_{Y,n} &= E[\ (Y - E[Y])^n\ ] \\
&= E[\ (aX + b - E[aX + b])^n\ ] \\
&= E[\ (aX + b - aE[X] - b)^n\ ] \\
&= E[\ (aX - aE[X])^n\ ] \\
&= a^n E[\ (X - E[X])^n\ ] \\
&= a^n \mu_{X,n}
\end{aligned}$$

Problem 2

- Let $a < b$ be integers. Let X be an r.v. whose outcomes are the integers in the interval $[a, b]$, and X takes each outcome with equal probability.

a) What is the PMF of X?

b) What is the PMF of $Y = e^X$?

c) What is $E[Y]$?

Problem 2 - Solution

a) There are $b-a+1$ integer outcomes in the interval $[a,b]$. So each of these has probability $1/(b-a+1)$, i.e.,

$$p_X(x) = \begin{cases} \frac{1}{b-a+1} & x \in \{a, a+1, \ldots, b\} \\ 0 & \text{else} \end{cases}$$

b) $Y = e^X$ takes $b-a+1$ distinct values since X does. So

$$p_Y(y) = \begin{cases} \frac{1}{b-a+1} & y \in \{e^a, e^{a+1}, \ldots, e^b\} \\ 0 & \text{else} \end{cases}$$

c)

$$\begin{aligned} E[Y] &= E[e^X] \\ &= \sum_{x=a}^{b} p_X(x) e^x \\ &= \frac{1}{b-a+1} \sum_{x=a}^{b} e^x \\ &= \frac{1}{b-a+1}(e^a + e^{a+1} + \cdots + e^b) \quad = \frac{e^a}{b-a+1}(1 + e + \cdots + e^{b-a}) \quad = \frac{e^a}{b-a+1} \frac{1 - e^{b-a+1}}{1-e} \end{aligned}$$

Problem 3

Consider the random variable X with PMF below. What is the PMF of $Y = |X - 1|$?

$$p_X(-2) = 0.1$$
$$p_X(-1) = 0.2$$
$$p_X(0) = 0.1$$
$$p_X(1) = 0.1$$
$$p_X(2) = 0.3$$
$$p_X(3) = 0.2$$

Problem 3 - Solution

Since $Y = |X-1|$ then Y can take outcomes $0, 1, 2$ and 3.

$P[Y=0] = P[|X-1|=0] = P[X=1] = 0.1$

$P[Y=1] = P[|X-1|=1] = P[\{X=0\} \cup \{X=2\}] = 0.1 + 0.3$

$P[Y=2] = P[|X-1|=2] = P[\{X=-1\} \cup \{X=3\}] = 0.2 + 0.2$

$P[Y=3] = P[|X-1|=3] = P[\{X=-2\} \cup \{X=4\}] = 0.1$

Problem 4

- Let N be the number of calls received at a call center over a period of t seconds.

a) What distribution do you think would be a good model for N and why?

b) Now assume that N is Poisson with parameter $\lambda = \alpha t$ where α is the average number of calls per second. Say $\alpha = 0.2$ calls/sec:

i) What is the probability of more than 3 calls over a period of 10 seconds?

ii) What is the probability of exactly 12 calls in one minute?

Problem 4 - Solution

a) There are large number of potential callers, but each has a small probability of calling over the period of t seconds, and the product of these two is a moderate number. So a Poisson distribution is likely a good model.

$$p_X(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda}, & \text{for } k = 0, 1, 2, \ldots \\ 0, & \text{else} \end{cases}$$

b) i) Here, N is Poisson with parameter $\lambda = 0.2 \times 10 = 2.0$.

$$\begin{aligned} P[N > 3] &= 1 - P[N = 0] - P[N = 1] - P[N = 2] - P[N = 3] \\ &= 1 - \frac{\lambda^0}{0!} e^{-\lambda} - \frac{\lambda^1}{1!} e^{-\lambda} - \frac{\lambda^2}{2!} e^{-\lambda} - \frac{\lambda^3}{3!} e^{-\lambda} \\ &= 1 - \frac{1}{1} e^{-2} - \frac{2}{1} e^{-2} - \frac{2^2}{2} e^{-2} - \frac{2^3}{6} e^{-2} \\ &= 1 - \frac{19}{3} e^{-2} \end{aligned}$$

ii) Here, N is Poisson with parameter $\lambda = 0.2 \times 60 = 12$.

$$\begin{aligned} P[N = 12] &= \frac{12^{12}}{12!} e^{-12} \\ &\approx 0.11437 \end{aligned}$$

Problem 5

- A manufacturer knows that 5% of its products are defective. Each day, an inspector randomly samples 20 products from a production batch. If the inspector finds more than 2 defective products, the entire batch is rejected. What is the probability that a batch passes the inspection?

Problem 5 - Solution

- we want *P(X ≤ 2)* and X~Binomial(n = 20, p = 0.05)

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$P(X = 0) = \binom{20}{0}(0.05)^0(0.95)^{20} \approx 0.3585$$

$$P(X = 1) = \binom{20}{1}(0.05)^1(0.95)^{19} = 20 \cdot 0.05 \cdot 0.95^{19} \approx 0.3774$$

$$P(X = 2) = \binom{20}{2}(0.05)^2(0.95)^{18} = 190 \cdot 0.0025 \cdot 0.95^{18} \approx 0.1887$$

$$\Rightarrow P(X \leq 2) \approx 0.3585 + 0.3774 + 0.1887 = 0.9246$$