# ECE 203
# Probability Theory and Statistics I
# Tutorial 6

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# Problem 1

You have an urn with 3 balls: one red, one green, and one blue. You keep drawing balls until you draw the green one. Let $X$ be the number of draws.

What is the PMF of $X$? What is $E[X]$?

# Problem 1 - Solution

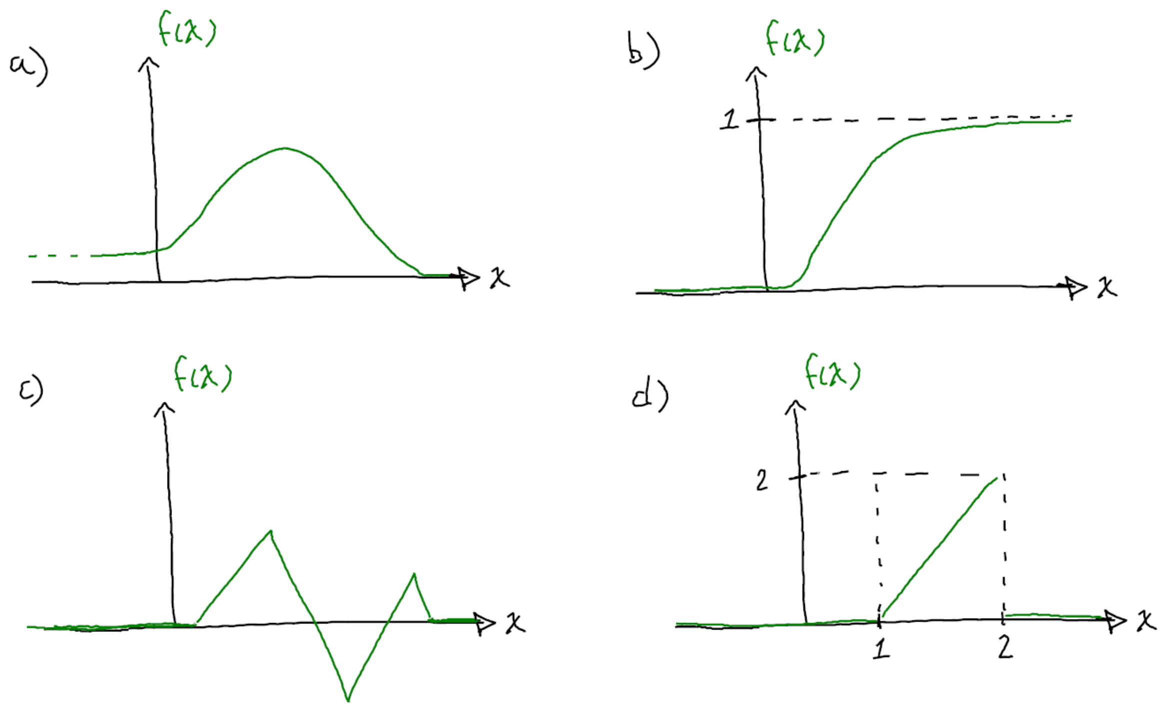
- We recognize this as a geometric random variable. If we draw the green ball, we have a “success”, and this occurs with probability 1/3 at each draw. Otherwise, we have a “failure,” and this happens with probability 2/3 at each draw.

$$X \sim \text{Geometric}(1/3)$$

If $X \sim \text{Geometric}(p)$ then $E[X] = 1/p$, so $E[X] = 3$.

# Problem 2

- Consider the following functions f(x). Can f(x) be the pdf of a random variable? If so, why? If not, why not?



# Problem 2 - Solution

a) Not a pdf. This is because the pdf should integrate to 1:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

However, because $f(x)$ goes to a positive constant as $x \to -\infty$, the area under the curve is infinite.

b) Not a pdf. Similar to a), the area under the curve is infinite, but should be 1. Note that this is a valid CDF though.

# Problem 2 - Solution

c) Not a pdf. While there is not enough information provided to tell if $\int_{-\infty}^{\infty} f(x)dx = 1$ or not, even if it is equal to 1, this is not a valid pdf. We can argue this two ways:

First, let $a < x < b$ be such that over this interval $f(x) < 0$. Then

$$P[X \in (a,b)] = \int_a^b f(x)dx < 0$$

so the probablity $P[X \in (a,b)]$ is -ve, which is not possible. So $f(x)$ is not a valid pdf.

A second way to see the problem is that a pdf can be found by differentiating the CDF

$$f_X(x) = \frac{d}{dx} F_X(x)$$

and since the cdf $F_X(x)$ is non-decreasing, then we should have $f_X(x) \geq 0$.

d) This $f(x)$ is non-negative, and integrates to 1, so it is a valid pdf.

# Problem 3

Suppose I have two functions $g(x)$ and $h(x)$ such that $g(x) \geq h(x)$ for all $x$.

Let $X$ be a r.v. with pdf $f_X(x)$. Show that

$$E[g(X)] \geq E[h(X)]$$

# Problem 3 - Solution

$$
\begin{aligned}
E[g(X)] &= \int_{-\infty}^{\infty} f_X(x) g(x) dx \\
&\geq \int_{-\infty}^{\infty} f_X(x) h(x) dx \qquad \text{since } f_X(x) \geq 0 \text{ and } g(x) \geq h(x) \\
&= E[h(X)]
\end{aligned}
$$

# Problem 4

A rv $X$ has a Cauchy distribution with mean 0 and scale parameter $\gamma > 0$ if its pdf is

$$f(x) = \frac{1}{\pi\gamma}\frac{1}{x^2 + \gamma^2}$$

a) Find $E[X^2]$

b) Find $E[X]$

c) Find $Var[X]$

# Problem 4 - Solution

a)

$$E[X^2] = \frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\frac{x^2}{x^2+\gamma^2}\,dx$$

$$= \infty \qquad \text{since } \frac{x^2}{x^2+\gamma^2} \to 1 \text{ as } x \to \pm\infty$$

b)

$$E[X] = \frac{1}{\pi\gamma}\int_{-\infty}^{\infty}\frac{x}{x^2+\gamma^2}\,dx$$

$$= \frac{1}{\pi\gamma}\int_{0}^{\infty}\frac{x}{x^2+\gamma^2}\,dx + \frac{1}{\pi\gamma}\int_{-\infty}^{0}\frac{x}{x^2+\gamma^2}\,dx$$

$$= \infty - \infty$$

$$= \text{undefined}$$

# Problem 4 - Solution

c)

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \text{undefined}
\end{aligned}
$$

# Problem 5

The random variable $X$ of the life length of a certain kind of battery (in hundreds of hours) is equal to:

$$f(x) = \begin{cases} \frac{1}{2}e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the probability that the life of a given battery is less than 200 or greater than 400 hours.

(ii) (ii) Find the probability that a battery of this type lasts for 300 hours if we know that it has already been in use for 200 hours.

# Problem 5 - Solution

(i) Let $A$ denote the event that $X$ is less than 2 and $B$ the event that $X$ is greater than 4. Then, because $A$ and $B$ are mutually exclusive,

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \int_0^2 \frac{1}{2} e^{-x/2}\, dx + \int_4^\infty \frac{1}{2} e^{-x/2}\, dx \\ &= (1 - e^{-1}) + e^{-2} \\ &= 1 - 0.368 + 0.135 \\ &= 0.767 \end{aligned}$$

# Problem 5 - Solution

(ii) We are interested in $P(X > 3|X > 2)$ and, by the definition of conditional probability,

$$P(X > 3|X > 2) = \frac{P(X > 3)}{P(X > 2)}$$

because the intersection of the events $(X > 3)$ and $(X > 2)$ is the event $(X > 3)$. Now

$$\frac{P(X > 3)}{P(X > 2)} = \frac{\int_3^\infty \frac{1}{2} e^{-x/2}\, dx}{\int_2^\infty \frac{1}{2} e^{-x/2}\, dx} = \frac{e^{-3/2}}{e^{-1}} = e^{-1/2} = 0.606$$