ECE 203
Probability Theory and Statistics I
Tutorial 7

July 2025

UNIVERSITY OF WATERLOO | FACULTY OF ENGINEERING

Problem 1

Suppose a normal rv X is such that

$$P[X < 5] = P[X > 15] = \frac{1}{2}P[X < 10].$$

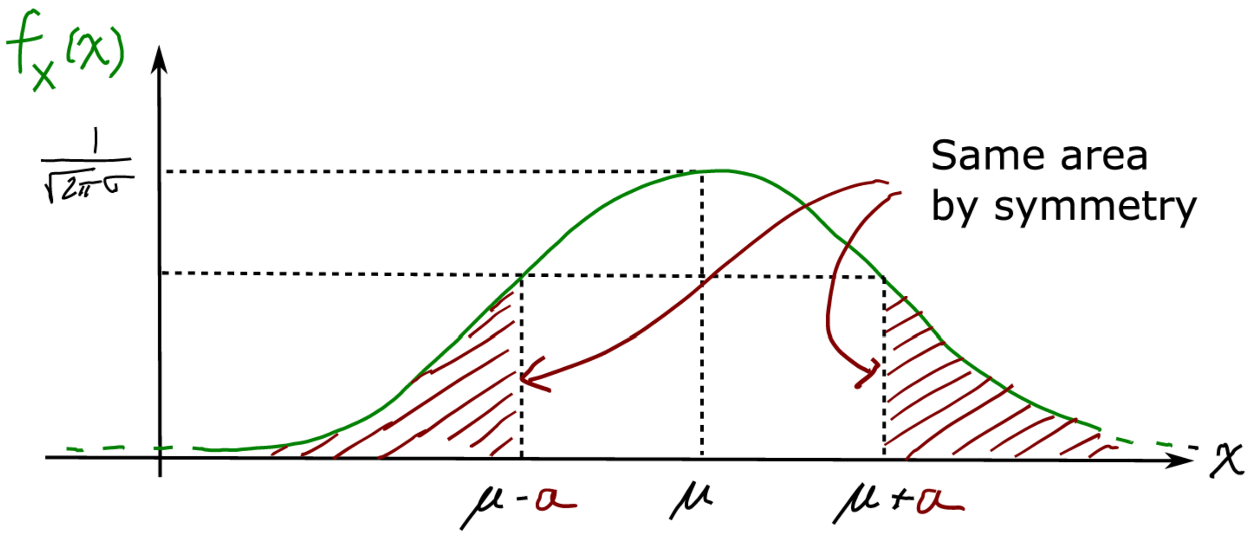
Find the distribution of X.

Problem 1 - Solution

First, note that since the pdf of a normal rv X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}},$$

the pdf is symmetric about its mean μ.



So, therefore, if μ is the mean of X, then

$$P[X < \mu - a] = P[X > \mu + a]$$

Problem 1 - Solution

$\mu - a = 5$

$\mu + a = 15$

and hence $\mu = 10$.

Also, $P[X < 10] = P[X < \mu] = 1/2$. Therefore $P[X > 15] = \frac{1}{2}P[X < 10] = 1/4$.

Finally,

$$P[X > 15] = P\left[\frac{X-\mu}{\sigma} > \frac{15-\mu}{\sigma}\right]$$
$$= 1 - \Phi\left(\frac{15-\mu}{\sigma}\right)$$
$$= 0.25$$

So, $\Phi(\frac{15-\mu}{\sigma}) = 0.75$. Using the Φ table, $\frac{15-\mu}{\sigma} = 0.67$

Since $\mu = 10$, then $\sigma = 7.46$

$X \sim \mathcal{N}(10, 7.46^2)$

Problem 2

Let $X \sim \mathcal{N}(0, 1)$ and $Z = \sqrt{|X|}$. What is the pdf of Z?

Problem 2 - Solution

Note that Z is non-negative, so $P[Z \leq a] = 0$ for $a < 0$. That leaves the case of $a \geq 0$:

$$\begin{aligned}
P[Z \leq a] &= P[\sqrt{|X|} \leq a] \\
&= P[|X| \leq a^2] \\
&= P[-a^2 \leq X \leq a^2] \\
&= 1 - P[X < -a^2] - P[X > a^2] \\
&= 1 - 2P[X < -a^2] \\
&= 1 - 2P[X \leq -a^2] \\
&= 1 - 2\Phi(-a^2)
\end{aligned}$$

Therefore, for $a \geq 0$:

$$\begin{aligned}
f_Z(a) &= \frac{d}{da} P[Z \leq a] \\
&= \frac{d}{da} \left(1 - 2\Phi(-a^2)\right) \\
&= \frac{d}{da} \left(1 - 2\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a^2} e^{-u^2/2} du\right) \\
&= \frac{-2}{\sqrt{2\pi}} e^{-a^4/2} \times -2a \\
&= \frac{2\sqrt{2}a}{\sqrt{\pi}} e^{-a^4/2}
\end{aligned}$$

Problem 2 - Solution

And, for $a < 0$:

$$f_Z(a) = \frac{d}{da} P[Z \leq a] = \frac{d}{da} 0 = 0$$

Problem 3

The time T for a server to process a job has pdf

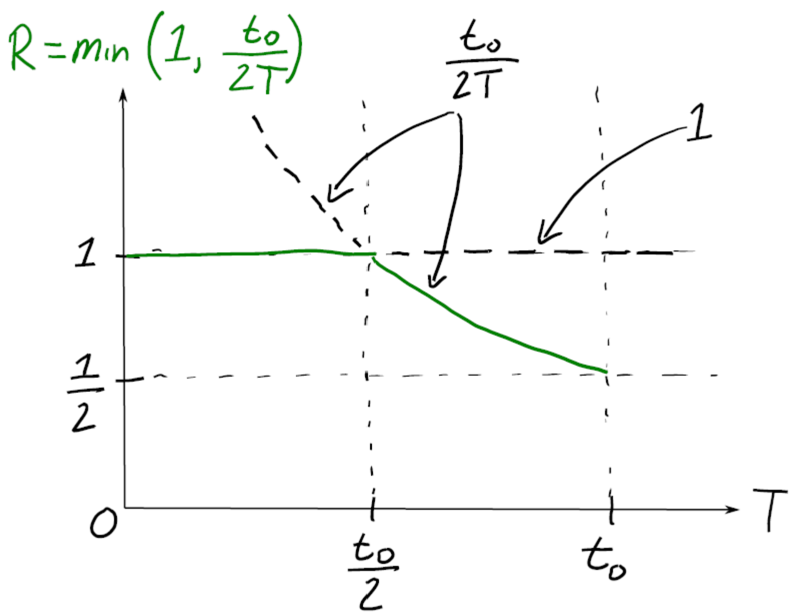
$$f_T(t) = \begin{cases} \frac{1}{t_0} & 0 < t < t_0 \\ 0 & \text{else} \end{cases}$$

The revenue from processing the job is $R = \min(1, \frac{t_0}{2T})$.

What is the cdf of R?

Problem 3 - Solution

Lets first plot the relationship between T and R:



Since $R = \min(1, \frac{t_0}{2T}) \leq 1$, then $F_R(r) = P[R \leq r] = 1$ for $r \geq 1$.

From the graph, since $0 < T < t_0$ then $\frac{1}{2} < R \leq 1$. So $F_R(r) = P[R \leq r] = 0$ for $r \leq 1/2$.

UNIVERSITY OF WATERLOO | FACULTY OF ENGINEERING

Problem 3 - Solution

So we now only consider that $1/2 < r < 1$. From the graph, for $1/2 < r < 1$ the events $\{R \leq r\}$ and $\{\frac{t_0}{2T} \leq r\}$ are the same:

$$\begin{aligned} F_R(r) &= P[R \leq r] \\ &= P\left[\frac{t_0}{2T} \leq r\right] \\ &= P\left[T \geq \frac{t_0}{2r}\right] \\ &= 1 - P\left[T < \frac{t_0}{2r}\right] \\ &= 1 - P\left[T \leq \frac{t_0}{2r}\right] \end{aligned}$$

Note that

$$\begin{aligned} P[T \leq t] &= \int_{-\infty}^{t} f_T(u)\, du \\ &= \begin{cases} 0 & t \leq 0 \\ t/t_0 & 0 < t < t_0 \\ 1 & t_0 \leq t \end{cases} \end{aligned}$$

Problem 3 - Solution

So, for $\frac{1}{2} < r < 1$ we have $1 < \frac{1}{r} < 2$ and $\frac{t_0}{2} < \frac{t_0}{2r} < t_0$. Therefore

$$\begin{aligned} F_R(r) &= 1 - P\left[T \leq \frac{t_0}{2r}\right] \\ &= 1 - \frac{t_0}{2r}\frac{1}{t_0} \\ &= 1 - \frac{1}{2r} \end{aligned}$$

Combining it all:

$$F_R(r) = \begin{cases} 0 & r \leq \frac{1}{2} \\ 1 - \frac{1}{2r} & \frac{1}{2} < r < 1 \\ 1 & 1 \leq r \end{cases}$$

Problem 4

Let the joint pdf of X and Y be given by

$$f_{XY}(x,y) = \begin{cases} c(1-xy) & 0 \leq x \leq 2, 0 \leq y \leq 2, xy \leq 1 \\ 0 & \text{else} \end{cases}$$

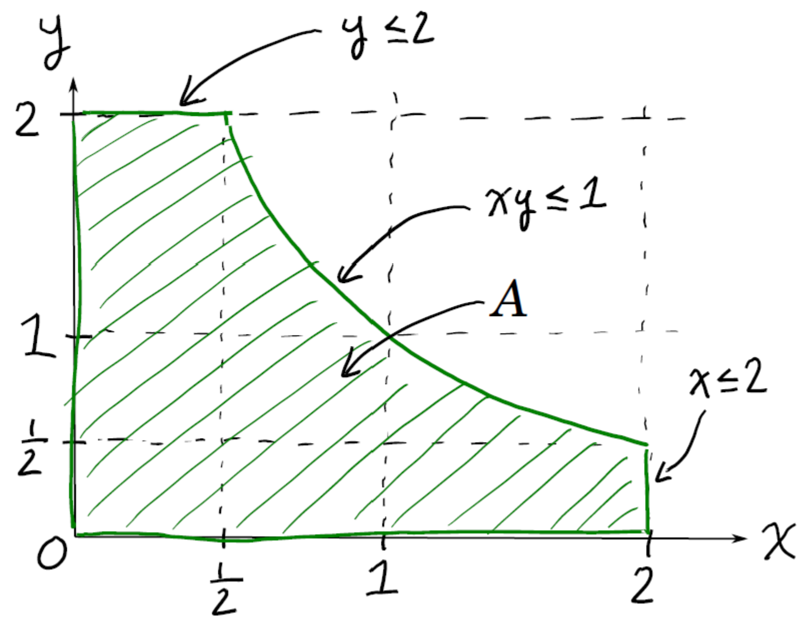
a) What is c?

b) What is $P[X > 1]$?

c) What is $P[X > Y]$?

Problem 4 - Solution

a) Sketching a figure is always a good idea:



$$
\begin{aligned}
1 &= \iint_A f_{XY}(x,y)dxdy \\
&= c\int_0^{1/2}\int_0^2 (1-xy)dydx + c\int_{1/2}^2\int_0^{1/x}(1-xy)dydx \\
&= c\int_0^{1/2}\left[y-\frac{xy^2}{2}\right]_{y=0}^{y=2}dx + c\int_{1/2}^2\left[y-\frac{xy^2}{2}\right]_{y=0}^{y=1/x}dx
\end{aligned}
$$

Problem 4 - Solution

$$= c\int_0^{1/2}[2-2x]dx + c\int_{1/2}^{2}\left[\frac{1}{x}-\frac{1}{2x}\right]dx$$
$$= c[2x - x^2]_0^{1/2} + \frac{c}{2}\left[\ln x\right]_{1/2}^{2}$$
$$= c \times \frac{3}{4} + c\ln 2$$

So $c = (3/4 + \ln 2)^{-1}$.

Problem 4 - Solution

b)

$$
\begin{aligned}
P[X > 1] &= \iint_{x>1} f_{XY}(x,y)dxdy \\
&= c\int_{x=1}^{2}\int_{y=0}^{1/x}(1-xy)dydx \\
&= c\int_{1}^{2}\left[y-\frac{xy^2}{2}\right]_{y=0}^{y=1/x} dx \\
&= c\int_{1}^{2}\left[\frac{1}{x}-\frac{1}{2x}\right]dx \\
&= \frac{c}{2}\left[\ln x\right]_1^2 \\
&= \frac{c}{2}\ln 2
\end{aligned}
$$

Problem 4 - Solution

c) Hard way:

$$
\begin{aligned}
P[X > Y] &= \iint_{x>y} f_{XY}(x,y)dxdy \\
&= \int_0^\infty \int_0^x f_{XY}(x,y)dydx \\
&= c\int_0^1 \int_0^x (1-xy)dydx + c\int_1^2 \int_0^{1/x} (1-xy)dydx \\
&= \cdots
\end{aligned}
$$

Easy way:

$$
\begin{aligned}
1 &= P[X > Y] + P[X = Y] + P[Y > X] \\
&= P[X > Y] + P[Y > X] && \text{since } x = y \text{ has 0 area in } x - y \text{ plane} \\
&= P[X > Y] + P[X > Y] && \text{by symmetry of pdf w.r.t. } x = y
\end{aligned}
$$

So $P[X > Y] = 1/2$.