ECE 203
Probability Theory and Statistics I
Tutorial 8

July 2025

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Problem 1

Let the joint pdf of X and Y be given by

$$f_{XY}(x,y) = \begin{cases} c(1-xy) & 0 \le x \le 2, 0 \le y \le 2, xy \le 1 \\ 0 & \text{else} \end{cases}$$

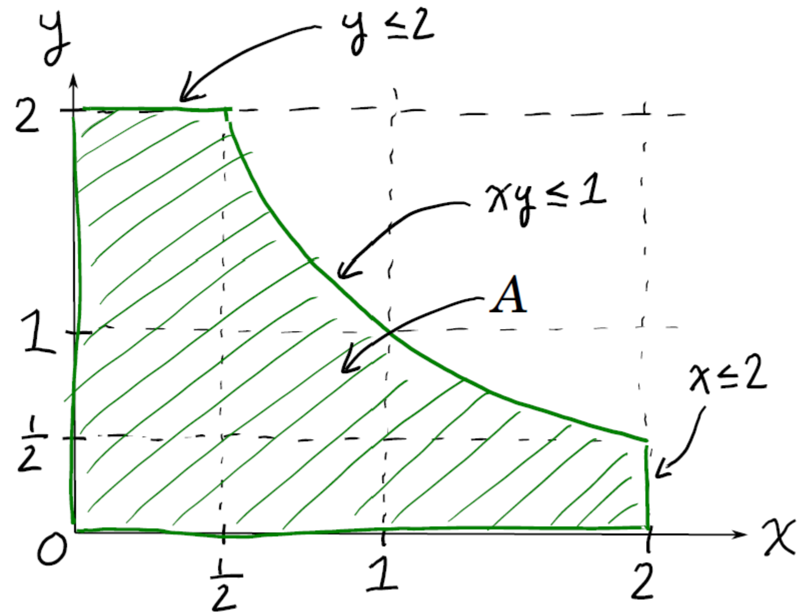
a) What is c?

b) What is $P[X > 1]$?

c) What is $P[X > Y]$?

Problem 1 - Solution

a) Sketching a figure is always a good idea:



$$1 = \iint_A f_{XY}(x,y)dxdy$$

$$= c\int_0^{1/2}\int_0^2 (1-xy)dydx + c\int_{1/2}^2\int_0^{1/x}(1-xy)dydx$$

$$= c\int_0^{1/2}\left[y - \frac{xy^2}{2}\right]_{y=0}^{y=2} dx + c\int_{1/2}^2\left[y - \frac{xy^2}{2}\right]_{y=0}^{y=1/x} dx$$

Problem 1 - Solution

$$= c\int_0^{1/2}[2-2x]dx + c\int_{1/2}^{2}\left[\frac{1}{x}-\frac{1}{2x}\right]dx$$

$$= c[2x-x^2]_0^{1/2} + \frac{c}{2}[\ln x]_{1/2}^{2}$$

$$= c\times\frac{3}{4} + c\ln 2$$

So $c = (3/4 + \ln 2)^{-1}$.

Problem 1 - Solution

b)

$$
\begin{aligned}
P[X > 1] &= \iint_{x>1} f_{XY}(x,y)\,dx\,dy \\
&= c\int_{x=1}^{2}\int_{y=0}^{1/x}(1-xy)\,dy\,dx \\
&= c\int_{1}^{2}\left[y-\frac{xy^2}{2}\right]_{y=0}^{y=1/x} dx \\
&= c\int_{1}^{2}\left[\frac{1}{x}-\frac{1}{2x}\right] dx \\
&= \frac{c}{2}\left[\ln x\right]_1^2 \\
&= \frac{c}{2}\ln 2
\end{aligned}
$$

Problem 1 - Solution

c) Hard way:

$$
\begin{aligned}
P[X > Y] &= \iint_{x>y} f_{XY}(x,y)dxdy \\
&= \int_0^{\infty}\int_0^{x} f_{XY}(x,y)dydx \\
&= c\int_0^{1}\int_0^{x}(1-xy)dydx + c\int_1^{2}\int_0^{1/x}(1-xy)dydx \\
&= \cdots
\end{aligned}
$$

Easy way:

$$
\begin{aligned}
1 &= P[X > Y] + P[X = Y] + P[Y > X] \\
&= P[X > Y] + P[Y > X] && \text{since } x = y \text{ has 0 area in } x - y \text{ plane} \\
&= P[X > Y] + P[X > Y] && \text{by symmetry of pdf w.r.t. } x = y
\end{aligned}
$$

So $P[X > Y] = 1/2$.

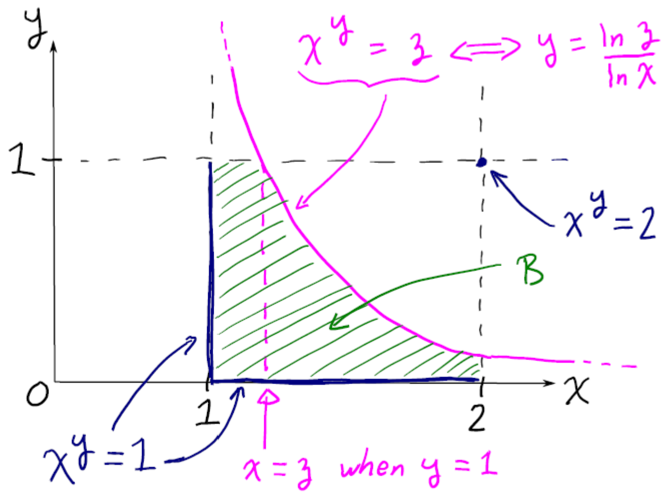
Problem 2

Let $X \sim U(1,2)$ and $Y \sim U(0,1)$ be independent rvs. Let

$$Z = X^Y$$

What is the pdf of Z?

Problem 2 - Solution



The joint pdf of X and Y is

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = 1 \cdot 1 = 1 \quad \text{on } [1,2] \times [0,1]$$

Since they are independent

$$f_{XY}(x,y) = \begin{cases} 1 & 1 \leq x \leq 2, 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$$

Problem 2 - Solution

Note that Z must be between 1 (when $Y = 0$ or $X = 1$) and 2 (when $X = 2, Y = 1$). So

$$F_Z(z) = \begin{cases} 0 & z \leq 1 \\ 1 & z \geq 2 \end{cases}$$

This leaves the case that $1 < z < 2$:

$$\begin{aligned} P[Z \leq z] &= P[X^Y \leq z] \\ &= P[Y \ln X \leq \ln z] \\ &= P[Y \leq \frac{\ln z}{\ln X}] \end{aligned}$$

Let

$$A = \{(x, y) \in \mathbb{R}^2 |\ 1 \leq x \leq 2, 0 \leq y \leq 1\}$$
$$B = \left\{(x, y) \in A \mid y \leq \frac{\ln z}{\ln x}\right\}$$

$$\begin{aligned} P[Z \leq z] &= \iint_{(x,y)\in B} f_{XY}(x, y) dx dy \\ &= \int_1^z \int_0^1 1 dy dx + \int_z^2 \int_0^{\frac{\ln z}{\ln x}} 1 dy dx \\ &= z - 1 + \int_z^2 \frac{\ln z}{\ln x} dx \end{aligned}$$

This integral can't be simplified further. To get the pdf,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 0 & z \leq 1 \\ 0 & z \geq 2 \end{cases}$$

Problem 2 - Solution

For $1 < z < 2$:

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz} F_Z(z) \\
&= \frac{d}{dz}\left[z - 1 + \int_z^2 \frac{\ln z}{\ln x} dx\right] \\
&= 1 + \frac{d}{dz}\int_z^2 \frac{\ln z}{\ln x} dx \\
&= 1 - \frac{\ln z}{\ln z} \times \frac{d}{dz} z + \int_z^2 \frac{d}{dz}\frac{\ln z}{\ln x} dx \\
&= \frac{1}{z}\int_z^2 \frac{1}{\ln x} dx
\end{aligned}
$$

by Leibniz's rule

$$
\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x, b(x)\big) \cdot \frac{d}{dx} b(x) - f\big(x, a(x)\big) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt
$$

Problem 3

Let X and Y be the outcomes of rolling a fair 4-sided and a fair 6-sided die. The rolls are independent. Let $Z = X + Y$. What is the PMF of Z?

Problem 3 - Solution

$$p_X(k) = \begin{cases} 1/4 & k = 1, \ldots, 4 \\ 0 & \text{else} \end{cases}$$

$$p_Y(k) = \begin{cases} 1/6 & k = 1, \ldots, 6 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} P[Z = n] &= P[X + Y = n] \\ &= P[\cup_{k=-\infty}^{\infty} \{X = k, Y = n - k\}] \\ &= \sum_{k=-\infty}^{\infty} P[X = k, Y = n - k] \\ &= \sum_{k=-\infty}^{\infty} P[X = k] P[Y = n - k] \\ &= \sum_{k=1}^{4} P[X = k] P[Y = n - k] \qquad \text{[since } X \in \{1, 2, 3, 4\}\text{]} \end{aligned}$$

Problem 3 - Solution

$$
\begin{aligned}
&= P[X = 1]P[Y = n - 1] \\
&\quad + P[X = 2]P[Y = n - 2] \\
&\quad + P[X = 3]P[Y = n - 3] \\
&\quad + P[X = 4]P[Y = n - 4] \\
&= \frac{1}{4}P[Y = n - 1] + \frac{1}{4}P[Y = n - 2] \\
&\quad + \frac{1}{4}P[Y = n - 3] + \frac{1}{4}P[Y = n - 4]
\end{aligned}
$$

Problem 3 - Solution

If $n \leq 1$, then sum is 0.

If $n = 2$, then sum is $\frac{1}{4}\frac{1}{6} = \frac{1}{24}$

If $n = 3$, then sum is $\frac{2}{24}$

If $n = 4$, then sum is $\frac{3}{24}$

If $n = 5$, then sum is $\frac{4}{24}$

If $n = 6$, then sum is $\frac{4}{24}$

If $n = 7$, then sum is $\frac{4}{24}$

If $n = 8$, then sum is $\frac{3}{24}$

If $n = 9$, then sum is $\frac{2}{24}$

If $n = 10$, then sum is $\frac{1}{24}$

If $n \geq 11$, then sum is 0

Problem 4

Let $X_1, \ldots, X_n$ are independent with $X_k \sim \mathcal{N}(\mu_k, \sigma_k^2)$.

a) If $Z_1 = a_1 X_1 + b_1, \ldots, Z_n = a_n X_n + b_n$, show that $Z_1, \ldots, Z_n$ are independent. [Assume $a_1 > 0, \ldots, a_n > 0$].

b) What is the pdf of $Z = (a_1 X_1 + b_1) + \cdots + (a_n X_n + b_n)$?

Problem 4 - Solution

a) Since $X_1, \ldots, X_n$ are independent:

$$F_{X_1,\ldots,X_n}(x_1, \ldots x_n) = F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)$$

Let $Y \sim \mathcal{N}(0,1)$. Since $X_k \sim \mathcal{N}(\mu_1, \sigma_1^2)$:

$$\begin{aligned} F_{X_k}(x_k) &= P[X_k \leq x_k] \\ &= P\left[\frac{X_k - \mu_k}{\sigma_k} \leq \frac{x_k - \mu_k}{\sigma_k}\right] \\ &= P\left[Y \leq \frac{x_k - \mu_k}{\sigma_k}\right] \\ &= \Phi\left(\frac{x_k - \mu_k}{\sigma_k}\right) \end{aligned}$$

So

$$F_{X_1,\ldots,X_n}(x_1, \ldots x_n) = \Phi\left(\frac{x_1 - \mu_1}{\sigma_1}\right) \times \cdots \times \Phi\left(\frac{x_n - \mu_n}{\sigma_n}\right)$$

Problem 4 - Solution

$$
\begin{aligned}
F_{Z_1,\ldots,Z_n}(z_1,\ldots z_n) &= P[Z_1 \le z_1,\ldots,Z_n \le z_n] \\
&= P[a_1X_1 + b_1 \le z_1,\ldots,a_nX_n + b_n \le z_n] \\
&= P\left[X_1 \le \frac{z_1 - b_1}{a_1},\ldots,X_n \le \frac{z_n - b_n}{a_n}\right] \\
&= F_{X_1,\ldots,X_n}\left(\frac{z_1 - b_1}{a_1},\ldots,\frac{z_n - b_n}{a_n}\right) \\
&= \Phi\left(\frac{\frac{z_1-b_1}{a_1} - \mu_1}{\sigma_1}\right) \times \cdots \times \Phi\left(\frac{\frac{z_n-b_n}{a_n} - \mu_n}{\sigma_n}\right) \\
&= \Phi\left(\frac{z_1 - b_1 - a_1\mu_1}{a_1\sigma_1}\right) \times \cdots \times \Phi\left(\frac{z_n - b_n - a_n\mu_n}{a_n\sigma_n}\right)
\end{aligned}
$$

So $Z_1,\ldots,Z_n$ are independent with $Z_k \sim \mathcal{N}(b_k + a_k\mu_k, a_k^2\sigma_k^2)$.

Problem 4 - Solution

b) Since $Z = Z_1 + \cdots + Z_n$ and Z_k's are Normal and independent, we apply Proposition : Z is Normal with

$$\mu_Z = E[Z_1] + \cdots + E[Z_n] = \sum_{k=1}^{n} b_k + a_k \mu_k$$

$$\sigma_Z^2 = Var[Z_1] + \cdots + Var[Z_n] = \sum_{k=1}^{n} a_k^2 \sigma_k^2$$

Recall that Proposition states that if $X_1, \ldots, X_n$ be independent with $X_k \sim \mathcal{N}(\mu_k, \sigma_k^2)$, then:
$X = X_1 + \cdots + X_n$ is $\sim \mathcal{N}(\mu_1 + \cdots + \mu_n, \sigma_1^2 + \cdots + \sigma_n^2)$.