# ECE 203
# Probability Theory and Statistics I
# Tutorial 9

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# Problem 1

Let $X_1, X_2, \ldots$ be iid with $E[X_1] = 2$ and $E[X_1^2] = 5$. Let $N \sim \text{Poisson}(\lambda)$ and independent of $X_1, X_2, \ldots$.

Find $E[X_1 X_2 \cdots X_N]$.

Find $Var[X_1 X_2 \cdots X_N]$.

# Problem 1 - Solution

We use conditional probability:

**Law of Total Expectation**

$$E[X_1 X_2 \cdots X_N] = E[\ E[X_1 X_2 \cdots X_N | N]\ ]$$

$$\begin{aligned} E[X_1 X_2 \cdots X_N | N = n] &= E[X_1 X_2 \cdots X_n | N = n] \\ &= E[X_1 X_2 \cdots X_n] \\ &= E[X_1] E[X_2] \cdots E[X_n] \\ &= 2 \times 2 \times \cdots \times 2 \\ &= 2^n \end{aligned}$$

$$\Rightarrow E[X_1 X_2 \cdots X_N | N] = 2^N$$

$$\begin{aligned} E[X_1 X_2 \cdots X_N] &= E[2^N] \\ &= e^{\lambda} \end{aligned}$$

$$\begin{aligned} E[a^N] &= \sum_{n=0}^{\infty} a^n P[N = n] \\ &= \sum_{n=0}^{\infty} a^n \frac{\lambda^n}{n!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(a\lambda)^n}{n!} \\ &= e^{-\lambda} e^{a\lambda} \\ &= e^{(a-1)\lambda} \end{aligned}$$

# Problem 1 - Solution

Approach 1: We use the Conditional Variance Formula (also called Law of Total Variance)

$$Var[X] = E[\,Var[X|Y]\,] + Var[\,E[X|Y]\,]$$

with $X = X_1 X_2 \cdots X_N$

$Y = N$

$$\begin{aligned} Var[\,E[X|Y]\,] &= Var[2^N] \\ &= E[(2^N)^2] - (E[2^N])^2 \\ &= E[4^N] - (E[2^N])^2 \\ &= e^{3\lambda} - (e^{\lambda})^2 \\ &= e^{3\lambda} - e^{2\lambda} \end{aligned}$$

$$\begin{aligned} &Var[X|Y] \\ &\quad = E[X^2|Y] - (\,E[X|Y]\,)^2 \\ &\quad = E[X_1^2 X_2^2 \cdots X_N^2|N] - (\,E[X_1 X_2 \cdots X_N|N]\,)^2 \\ &\quad = E[X_1^2 X_2^2 \cdots X_N^2|N] - (2^N)^2 \end{aligned}$$

# Problem 1 - Solution

$$
\begin{aligned}
E[X_1^2 X_2^2 \cdots X_N^2 | N = n] \\
&= E[X_1^2 X_2^2 \cdots X_n^2 | N = n] \\
&= E[X_1^2 X_2^2 \cdots X_n^2] \\
&= E[X_1^2] E[X_2^2] \cdots E[X_n^2] \\
&= 5^n
\end{aligned}
$$

$$\Rightarrow E[X_1^2 X_2^2 \cdots X_N^2 | N] = 5^N$$

$$
\begin{aligned}
\Rightarrow E[\ Var[X|Y]\ ] \\
&= E[5^N - 4^N] \\
&= e^{4\lambda} - e^{3\lambda}
\end{aligned}
$$

Combining

$$Var[X] = e^{4\lambda} - e^{3\lambda} + (e^{3\lambda} - e^{2\lambda}) = e^{4\lambda} - e^{2\lambda}$$

# Problem 1 - Solution

Approach 2: More direct calculation

$$Var[X] = E[X^2] - (E[X])^2$$

$$\begin{aligned} E[X] &= E[\ E[X|Y]\ ] \\ &= E[\ E[X_1 X_2 \cdots X_N|N]\ ] \\ &= E[\ 2^N\ ] \\ &= e^{\lambda} \end{aligned}$$

$$\begin{aligned} E[X^2] &= E[\ E[X^2|Y]\ ] \\ &= E[\ E[X_1^2 X_2^2 \cdots X_N^2|N]\ ] \\ &= E[\ 5^N\ ] \\ &= e^{4\lambda} \end{aligned}$$

$$\Rightarrow Var[X] = e^{4\lambda} - (e^{\lambda})^2$$

# Problem 2

Let $(X, Y)$ be jointly Gaussian with mean vector $\boldsymbol{\mu} = 0$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

Let

$$U = X + 2Y$$
$$V = 2X + Y$$

a) What are $Cov[X, Y]$ and $E[XY]$?

b) What is the correlation $\rho(U, V)$?

# Problem 2 - Solution

a)

$$\rho = \rho(X,Y) = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} = \frac{E[XY] - \overbrace{E[X]}^{=0}\,\overbrace{E[Y]}^{=0}}{\sigma_X \sigma_Y}$$

$$\Rightarrow Cov[X,Y] = \rho \sigma_X \sigma_Y$$

$$E[XY] = \rho \sigma_X \sigma_Y$$

b) We want

$$\rho(U,V) = \frac{Cov[U,V]}{\sqrt{Var[U]Var[V]}}$$

# Problem 2 - Solution

$$\begin{aligned}Cov[U,V] &= E[UV] - E[U]E[V] \\ &= E[(X+2Y)(2X+Y)] - E[X+2Y]E[Y+2X] \\ &= E[2X^2+5XY+2Y^2] - 0\times 0 \\ &= 2E[X^2]+5E[XY]+2E[Y^2] \\ &= 2\sigma_X^2 + 5\rho\sigma_X\sigma_Y + 2\sigma_Y^2\end{aligned}$$

$$\begin{aligned}Var[U] &= E[U^2] - (E[U])^2 \\ &= E[(X+2Y)^2] - (E[X+2Y])^2 \\ &= E[X^2+4XY+4Y^2] - 0^2 \\ &= \sigma_X^2 + 4\rho\sigma_X\sigma_Y + 4\sigma_Y^2\end{aligned}$$

$$\begin{aligned}Var[V] &= E[V^2] - (E[V])^2 \\ &= E[(2X+Y)^2] - (E[2X+Y])^2 \\ &= E[4X^2+4XY+Y^2] - 0^2 \\ &= 4\sigma_X^2 + 4\rho\sigma_X\sigma_Y + \sigma_Y^2\end{aligned}$$

# Problem 3

Let $X_1, X_2, \ldots, X_n$ be independent and $\sim \mathcal{N}(0, 1)$. Let

$$Y = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$
$$Z = b_1 X_1 + b_2 X_2 + \cdots + b_n X_n$$

Find $Cov[Y, Z]$.

# Problem 3 - Solution

$$
\begin{aligned}
Cov[Y, Z] &= Cov\left[\sum_{k=1}^{n} a_k X_k, \sum_{m=1}^{n} b_m X_m\right] \\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} Cov[a_k X_k, b_m X_m] \\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} a_k Cov[X_k, b_m X_m] \\
&= \sum_{k=1}^{n}\sum_{m=1}^{n} a_k b_m Cov[X_k, X_m]
\end{aligned}
$$

Since $X_1, X_2, \ldots, X_n$ are independent, $Cov[X_k, X_m] = 0$ when $k \neq m$:

$$
\begin{aligned}
Cov[Y, Z] &= \sum_{k=1}^{n}\left[\sum_{m=1}^{n} a_k b_m Cov[X_k, X_m]\right] \\
&= \sum_{k=1}^{n}\left[\sum_{m=k}^{k} a_k b_m Cov[X_k, X_m]\right]
\end{aligned}
$$

# Problem 3 - Solution

$$= \sum_{k=1}^{n} a_k b_k Cov[X_k, X_k]$$

$$= \sum_{k=1}^{n} a_k b_k Var[X_k]$$

$$= \sum_{k=1}^{n} a_k b_k$$

# Problem 4

Two particles are positioned on a line with random and independent positions

$$X_1 \sim \mathcal{N}(0, 1)$$
$$X_2 \sim U(-1, 1)$$

Find $E[(X_1 + X_2)^3]$.

# Problem 4 - Solution

$$
\begin{aligned}
E[(X_1 + X_2)^3] &= E[X_1^3 + 3X_1^2 X_2 + 3X_1^1 X_2^2 + X_2^3] \\
&= E[X_1^3] + 3E[X_1^2 X_2] + 3[X_1^1 X_2^2] + E[X_2^3] \\
&= E[X_1^3] + 3E[X_1^2]E[X_2] + 3E[X_1^1]E[X_2^2] + E[X_2^3] \quad \text{(a)} \\
&= E[X_1^3] + 3E[X_1^2] \times 0 + 3 \times 0 \times E[X_2^2] + E[X_2^3] \\
&= E[X_1^3] + E[X_2^3] \\
&= 0 \quad \text{(b)}
\end{aligned}
$$

(a) follows because if $X_1$ and $X_2$ are independent, then $E[f(X_1)g(X_2)] = E[f(X_1)]E[g(X_2)]$

(b) follows from

$$
\begin{aligned}
E[X_2^3] &= \int_{-1}^{1} x^3 \times \frac{1}{2} dx \\
&= 0 \quad \text{since integrand is an odd function}
\end{aligned}
$$

# Problem 4 - Solution

$$E[X_1^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-x^2/2} dx$$

$$= 0 \qquad \text{since integrand is an odd function}$$