Chapter 1

Introduction to Signals and Systems

The goal of this chapter is to introduce the concepts of signal and system, and to provide a foundation for the later chapters of this course, in which we look at these concepts in considerable detail.

1.1 Signals

Informally a signal is a quantity that varies with time.

Example 1.1.1 Typical examples of signals:

(a) The voltage at the terminals of a transmission or receiving antenna.
(b) The atmospheric pressure at a given location on the earth.
(c) The monthly reports on the national rate of inflation.
(d) The output voltage of an analogue-to-digital converter in a C.D. player.

In Example 1.1.1(a), (b), the relevant quantity (voltage, atmospheric pressure) is given for all values of time \( t \), and these signals are called \textit{continuous-time signals}. In Example 1.1.1(c), (d), the relevant quantity (rate of inflation, voltage) is given at only \textit{discrete} instants of time \( t \), and these signals are called \textit{discrete-time signals}. We can now formalize these notions as follows:

Definition 1.1.2 A \textit{continuous-time signal} is a real or complex valued function defined for all values of \( t \) over the range \(-\infty < t < +\infty\). See Fig. 1.1.
Definition 1.1.3 A discrete-time signal is a real or complex valued function defined over a discrete set of instants \( \{ t_k \} \). Throughout this course we shall assume that \( t_k \) is of the form \( t_k = k\gamma \), \( k = \ldots, -2, -1, 0, 1, 2, \ldots \), where \( \gamma > 0 \) is a constant called the intersample interval. See Fig. 1.1.

Remark 1.1.4 We write \( x(t) \) for the value of a continuous-time signal at instant \( t \), and write \( x[k] \) (rather than \( x(k\gamma) \)) for the value of a discrete-time signal at instant \( k\gamma \).

Remark 1.1.5 Perhaps the very simplest signal is the zero signal whose value is identically zero over the whole range of definition. That is, a continuous-time signal \( x(t) \) is called a zero signal when \( x(t) = 0 \) for all \( t \), and a discrete-time signal \( x[k] \) is called a zero signal when \( x[k] = 0 \) for all \( k \).

We are now going to introduce certain standard signals (besides the zero signal) which will be essential for this course. These signals are:

- Step signals
- Periodic signals
- Sinusoidal signals
- Generalized sinusoidal signals
1.1.1 Step Signals

Definition 1.1.6 A continuous-time step signal $x(t)$ has the form

\[ x(t) = \begin{cases} 
0 & \text{when } t < 0, \\
A & \text{when } t \geq 0,
\end{cases} \]

where $A$ is a real or complex constant. See Fig. 1.3. In the special case of $A = 1$ we call this signal a unit step function and denote it by $u(t)$.

The discrete-time analogue is formulated in the same way:

Definition 1.1.7 A discrete-time step signal $x[k]$ has the form

\[ x[k] = \begin{cases} 
0 & \text{when } k < 0, \\
A & \text{when } k \geq 0,
\end{cases} \]

where $A$ is a real or complex constant. See Fig. 1.4. In the special case of $A = 1$ we call this signal a unit step function and denote it by $u[k]$. 
1.1.2 Periodic Signals

Periodic signals are essentially repetitive signals. We make this idea precise as follows:

Definition 1.1.8 A continuous-time signal $x(t)$ is called periodic when the following relation holds for some constant $T > 0$:

$$x(t) = x(t + T), \text{ for all real } t.$$  \hfill (1.1)

The constant $T$ is called a period of the signal. See Fig. 1.5.

![Diagram](image)

Remark 1.1.9 If $x(t)$ is periodic with a period $T$ then clearly

$$x(t) = x(t + T) = x(t + 2T) = \ldots = x(t + nT), \text{ for all real } t,$$

for each positive integer $n$. Thus, if a continuous-time signal $x(t)$ is periodic with period $T$, then it is also periodic with periods $2T, 3T, 4T, \ldots$. The smallest period is called the fundamental period of $x(t)$.

An important question is the following: Is the sum of two continuous-time periodic signals also periodic! In general the answer is NO, but we do have the following result:

Fact 1.1.10 Suppose that $x_1(t)$ is a continuous-time periodic signal with period $T_1$, and $x_2(t)$ is a continuous-time periodic signal with period $T_2$. If $T_1/T_2$ is a rational number (i.e. the ratio of two integers) then $x(t) \equiv x_1(t) + x_2(t)$ is a continuous-time periodic signal.

Proof: Since $T_1/T_2$ is rational we can write

$$
\frac{T_1}{T_2} = \frac{n_1}{n_2}
$$

where $n_1$ and $n_2$ are integers, thus

$$n_2T_1 = n_1T_2. \hfill (1.2)$$
By Remark (1.1.9) we have

(1.3) \[ z_1(t) = z_1(t + n_1T_1), \quad \text{and} \quad z_2(t) = z_2(t + n_2T_2), \quad \text{for all real } t. \]

Let \( T \) denote the common value in (1.2). Then, adding the expressions in (1.3), gives

\[ z_1(t) + z_2(t) = z_1(t + T) + z_2(t + T), \quad \text{for all real } t, \]

or

\[ z(t) = z(t + T), \quad \text{for all real } t. \]

This shows that the sum \( z(t) \) is periodic with a period \( T \).

\[ \Box \]

**Remark 1.1.11** Let \( z(t) \) be continuous-time periodic signal with a period \( T \). A useful fact is that

(1.4) \[ \int_{t_0}^{t_0+T} z(t) \, dt = \int_{t_0}^{T} z(t) \, dt, \quad \text{for all } t_0. \]

To see this write

(1.5) \[ \int_{t_0}^{t_0+T} z(t) \, dt = \int_{t_0}^{T} z(t) \, dt + \int_{T}^{t_0+T} z(t) \, dt = \int_{t_0}^{t_0+T} z(t) \, dt. \]

Now, taking \( \tau = t - T \), gives

(1.6) \[ \int_{T}^{t_0+T} z(t) \, dt = \int_{0}^{t_0} z(\tau + T) \, d\tau. \]

Since \( z(t) \) is periodic with period \( T \), we have

\[ z(\tau + T) = z(\tau), \quad \text{for all } \tau, \]

thus

\[ \int_{0}^{t_0} z(\tau + T) \, d\tau = \int_{0}^{t_0} z(\tau) \, d\tau. \]

Thus, from (1.6), we get

(1.7) \[ \int_{T}^{t_0+T} z(t) \, dt = \int_{0}^{t_0} z(\tau) \, d\tau. \]

By (1.7) we can cancel the last two terms on the right side of (1.5) to get (1.4).

We can also formulate the notion of periodicity for a discrete-time signal:
Definition 1.1.12 A discrete-time signal \( x[k] \) is called periodic when the following relation holds for some positive integer constant \( T \):

\[
x[k] = x[k + T], \quad \text{for all integers } k.
\]

The constant \( T \) is called a period of the signal. See Fig. 1.6.

Remark 1.1.13 Exactly as in Remark 1.1.9 we see that when a discrete-time signal \( x[k] \) is periodic with a period \( T \), then it is also periodic with periods \( 2T, 3T, \ldots \). The smallest period is called the fundamental period of \( x[k] \).

As in the continuous-time case, we can ask the following question: Is the sum of two discrete-time periodic signals also periodic? In this case the periods of the two signals are positive integers, hence their ratio is a rational number, and hence, exactly as in Fact 1.1.10, we can establish:

Fact 1.1.14 Suppose that \( x_1[k] \) is a discrete-time periodic signal with period \( T_1 \) and \( x_2[k] \) is a discrete-time periodic signal with period \( T_2 \). Then \( x[k] = x_1[k] + x_2[k] \) is a discrete-time periodic signal.

Thus, the sum of two discrete-time periodic signals is always periodic, in contrast to the continuous-time case, where the sum of periodic signals need not be periodic.
1.1.3 Sinusoidal Signals

A particularly important class of continuous-time periodic signals are the sinusoids, which have the following general form:

\[ x(t) = A \cos(\omega t + \theta). \]  

See Fig. 1.7. Here, \( A > 0 \) is a real constant called the amplitude; \( \omega \) is a real constant called the angular frequency, with units of radians/sec; and \( \theta \) is a real constant called the phase, with units of radians. Also, put

\[ f = \frac{\omega}{2\pi}, \quad T = \frac{1}{f}. \]

Then \( f \) is called the frequency with units of sec\(^{-1}\) (or Hertz); and \( T \) has the units of seconds. We see from (1.9) that

\[ x(t) = x(t + T), \quad \text{for all real } t, \]

so that \( x(t) \) is periodic with period \( T \). There is a useful geometric interpretation of \( x(t) \). Visualize the complex number \( Ae^{j\theta} \). If this complex number is rotated by \( \omega t \) in a counterclockwise direction, then \( x(t) \) is the projection of the resulting complex number onto the real axis. See Fig. 1.8.

The complex number \( Aej\theta \) is called the phasor of the sinusoidal signal (1.9). Phasors are not used much in this course, although they are extremely important for the analysis of a.c. circuits. Of much greater importance is the exponential form of the sinusoid (1.9). To establish this, recall the
this, recall the Euler identity

\[ e^{j\alpha} = \cos \alpha + j\sin \alpha, \] (1.10)

which holds for angles \( \alpha \) expressed in radians. From (1.10) we then get the standard identities

\[ \cos \alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}, \quad \sin \alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}, \] (1.11)

which again holds for angles \( \alpha \) expressed in radians. Then, from (1.9) and (1.11), we get

\[ z(t) = \frac{A}{2} [e^{-j\omega t} + e^{j\omega t}] + \frac{A}{2} e^{-j\phi} + \frac{A}{2} e^{j\phi} e^{-j\omega t} \]

thus

\[ z(t) = De^{j\omega t} + D^* e^{-j\omega t}, \] (1.12)

where

\[ D \triangleq \frac{A}{2} e^{j\phi} \quad \text{and} \quad D^* \triangleq \frac{A}{2} e^{-j\phi}, \] (1.13)

is the complex conjugate of \( D \). We call (1.12) the exponential form of the sinusoid (1.9). The usefulness of the exponential form (1.12) derives from the fact that the exponential functions in (1.12) are much easier to handle than the cosine function in (1.9).
1.1.4 Generalized Sinusoids

There is an important generalization of the sinusoid, called the generalized sinusoid, which has the following form:

\[ x(t) = Ae^{\sigma t} \cos(\omega t + \theta). \]  

(1.14)

See Fig. 1.9.

Fig. 1.9.

Here, \( A, \omega \) and \( \theta \) are just as in (1.9), and \( \sigma \) is a real constant. Clearly \( x(t) \) cannot be periodic unless \( \sigma = 0 \), in which case the generalized sinusoid (1.14) reduces to the sinusoid (1.9). Combining (1.14) and (1.12) we clearly get

\[ x(t) = e^{\sigma t} \left[ D e^{\omega t} + D^* e^{-\omega t} \right], \]

thus

\[ x(t) = De^{\sigma t} + D^* e^{-\sigma t}, \]  

(1.15)

where \( s \triangleq \sigma + j\omega \) is called the complex frequency, and \( D, D^* \) are given by (1.13). This is called the exponential form of the generalized sinusoid (1.14), and will be particularly useful when we look at Laplace transformations of continuous-time signals.
1.1.5 Energy and Average Power of Signals

Definition 1.1.15 Suppose that \( x(t) \) is a continuous-time signal. Then the quantity

\[
E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt
\]

is called the energy of the signal. When \( E < \infty \) then \( x(t) \) is called an energy signal.

A precisely analogous definition can be formulated in the discrete-time case:

Definition 1.1.16 Suppose that \( x[k] \) is a discrete-time signal. Then the quantity

\[
E \triangleq \sum_{k=-\infty}^{\infty} |x[k]|^2
\]

is called the energy of the signal. When \( E < \infty \) then \( x[k] \) is called an energy signal.

Remark 1.1.17 The quantity \( E \) often has very little to do with energy in the physical sense of the word. For example, if the discrete-time signal \( x[k] \) represents the monthly rate of inflation, then the quantity \( E \) in Definition 1.1.16 clearly has nothing to do with energy at all! Throughout this course, the word “energy” in the preceding definitions will be interpreted as a quantity which measures the size of a signal, and will usually not have any physical significance.

Example 1.1.18 Determine the energy of the following signals:

(i) \( x(t) = e^{-t} \), and (ii) \( x(t) = \begin{cases} 0 & \text{when } t < 0, \\ e^{-t} & \text{when } t \geq 0. \end{cases} \)

(i) Clearly

\[
E = \int_{-\infty}^{\infty} e^{-2t} \, dt = +\infty.
\]

(ii) Here

\[
E = \int_{0}^{\infty} e^{-2t} \, dt = \frac{1}{2}
\]

Definition 1.1.19 Suppose that \( x(t) \) is a continuous-time signal. Then the quantity

\[
P(a) \triangleq \frac{1}{2a} \int_{-a}^{a} |x(t)|^2 \, dt, \quad (a > 0),
\]

is called the average power of the signal.

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is the **power** of the signal over the interval $-a \leq t \leq a$. Moreover, if the limit

$$P \triangleq \lim_{a \to \infty} P(a)$$

exists, then $P$ is called the **average power** of the signal, and $x(t)$ is called a **power signal** when $0 < P < \infty$.

**Definition 1.1.29** Suppose that $x[k]$ is a discrete-time signal. Then the quantity

$$P(n) \triangleq \frac{1}{2n+1} \sum_{k=-n}^{n} |x[k]|^2, \quad (n = 1, 2, \ldots)$$

is the **power** of the signal over the interval $-n \leq k \leq n$. Moreover, if the limit

$$P \triangleq \lim_{n \to \infty} P(n)$$

exists, then $P$ is called the **average power** of the discrete-time signal, and $x[k]$ is called a **power signal** when $0 < P < \infty$.

**Example 1.1.21** Calculate the energy and average power of the continuous-time signal

$$x(t) = A u(t).$$

Then

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{0}^{\infty} A^2 \, dt = +\infty.$$ 

For the average power we have

$$P(a) = \frac{1}{2a} \int_{-a}^{a} |x(t)|^2 \, dt = \frac{1}{2a} \int_{0}^{a} A^2 \, dt = \frac{A^2}{2},$$

thus

$$P \triangleq \lim_{a \to \infty} P(a) = \frac{A^2}{2}.$$ 

**Example 1.1.22** Find the energy and average power of the continuous-time signal

$$x(t) = Ae^{-at} u(t),$$

where $A$ and $\alpha > 0$ are real constants. For the energy:

$$E = \int_{-\infty}^{\infty} x^2(t) \, dt = \int_{0}^{\infty} A^2 e^{-2\alpha t} \, dt = \frac{A^2}{2\alpha}.$$
For the average power we have
\[
P(a) = \frac{1}{2a} \int_{-a}^{a} x^2(t) \, dt = \frac{A^2}{2a} \int_{0}^{a} e^{-2\alpha t} \, dt = \frac{A^2}{4\alpha} \left[ 1 - e^{-2\alpha a} \right].
\]
Thus,
\[
P = \lim_{a \to \infty} \frac{A^2}{4\alpha} \left[ 1 - e^{-2\alpha a} \right] = 0.
\]

**Remark 1.1.23** The average power for a periodic continuous-time signal \(x(t)\) with period \(T\) is always given by
\[
P = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, dt.
\]
To see this, note that, since \(x(t)\) is periodic with period \(T\), it follows that \(|x(t)|^2\) is also periodic with period \(T\), hence periodic with period \(nT\), for each \(n = 1, 2, \ldots\) (recall Remark 1.1.9). Thus, from Remark 1.1.11, we have
\[
\int_{-\frac{nT}{2}}^{\frac{nT}{2}} |x(t)|^2 \, dt = \int_{0}^{T} |x(t)|^2 \, dt = \sum_{k=0}^{n} \int_{(k-1)T}^{kT} |x(t)|^2 \, dt = n \int_{0}^{T} |x(t)|^2 \, dt,
\]
where the last equality in (1.17) follows because \(|x(t)|^2\) is periodic with period \(T\), so that
\[
\int_{(k-1)T}^{kT} |x(t)|^2 \, dt = \int_{0}^{T} |x(t)|^2 \, dt, \quad k = 1, 2, \ldots, n.
\]
From (1.17) we get
\[
\frac{1}{nT} \int_{-\frac{nT}{2}}^{\frac{nT}{2}} |x(t)|^2 \, dt = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, dt, \quad n = 1, 2, \ldots,
\]
that is, for \(a(n) = \frac{n}{2} T/2\), one has
\[
\frac{1}{2a(n)} \int_{-a(n)}^{a(n)} |x(t)|^2 \, dt = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, dt, \quad n = 1, 2, \ldots,
\]
Since \(c(n) \to \infty\) as \(n \to \infty\), we see from Definition 1.1.19 and (1.18) that the average power is given by
\[
P = \lim_{n \to \infty} \frac{1}{2a(n)} \int_{-a(n)}^{a(n)} |x(t)|^2 \, dt = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, dt,
\]
as required.

**Example 1.1.24** Find the energy and power of the sinusoid
\[
x(t) = A \cos(\omega t + \theta).
\]
For the energy we have

\[ E = \int_{-\infty}^{\infty} x^2(t) \, dt = A^2 \int_{-\infty}^{\infty} \cos^2(\omega t + \theta) \, dt = +\infty. \]

For the average power, observe that the period of \( x(t) \) is \( T = 2\pi/\omega \). Thus, by Remark 1.1.23:

\[
P = \frac{1}{T} \int_{0}^{T} x^2(t) \, dt
= \frac{A^2 \omega}{2\pi} \int_{0}^{2\pi/\omega} \cos^2(\omega t + \theta) \, dt
= \frac{A^2}{2\pi} \int_{\theta}^{2\pi + \theta} \cos^2(\tau) \, d\tau \quad \text{for} \quad \tau = \omega t + \theta, \text{thus} \quad d\tau = \omega dt
= \frac{A^2}{2\pi} \int_{0}^{2\pi} \cos^2(\tau) \, d\tau
\]

where the final equality follows by Remark 1.1.11 and that fact that \( \cos^2(\tau) \) is periodic with a period \( 2\pi \). Now, recalling the useful identity

\[ \cos^2(\tau) = \frac{1}{2} [1 + \cos(2\tau)] \]

we get

\[
\int_{0}^{2\pi} \cos^2(\tau) \, d\tau = \frac{1}{2} \int_{0}^{2\pi} [1 + \cos(2\tau)] \, d\tau = \pi,
\]

where the second equality follows since

\[ \int_{0}^{2\pi} \cos(2\tau) \, d\tau = 0. \]

Putting (1.20) into (1.19) gives

\[ P = \frac{A^2}{2}. \]
1.1.6 Transformations of the Time Variable

Let \( z(t) \) be a continuous-time signal. We can transform this into a variety of other continuous-time signals as follows:

(a) Time Inversion: Put

\[
y(t) \triangleq z(-t).
\]

See Fig. 1.10.

(b) Time Scaling: For a real constant \( a > 0 \) put

\[
y(t) \triangleq z(at).
\]

See Fig. 1.11.

(c) Time Shift: For a real constant \( t_0 \) put

\[
y(t) \triangleq z(t - t_0).
\]

See Fig. 1.12.
**Remark 1.1.25** Suppose that \( x(t) \) is the audio signal on a tape-recording. Then \( y(t) \) given by the time inversion (a) is the signal when the tape is played backwards. If \( a = 2 \) in the time scaling (b) then \( y(t) \) is the signal when the tape is played forward at twice the normal speed, while if \( a = 1/2 \) then \( y(t) \) is the signal when the tape is played at half the normal speed.

One can formulate precisely analogous ideas for a discrete-time signal \( x[k] \):

(a) **Time Inversion:** Put
\[
y[k] \triangleq x[-k].
\]

(b) **Time scaling:** For an integer constant \( a > 0 \) put
\[
y[k] \triangleq x[an].
\]

(c) **Time shift:** For an integer constant \( k_0 \) put
\[
y[k] \triangleq x[k - k_0].
\]

Using transformations of the time variable enables us to use simple signals such as step functions to synthesize more complicated signals. The next examples show this:

**Example 1.1.26** Suppose that \( x(t) \) is given by
\[
x(t) = \begin{cases} 
2 & \text{when } 3 \leq t < 5, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus
\[
x(t) = 2[u(t - 3) - u(t - 5)].
\]
Example 1.1.27 Suppose that \( x(t) \) is given by

\[
x(t) = \begin{cases} 
1 & \text{when } t < -1, \\
0 & \text{otherwise.}
\end{cases}
\]

Then

\[ x(t) = 1 - u(t + 1). \]

Example 1.1.28 Suppose that \( x(t) \) is given by

\[
x(t) = \begin{cases} 
0 & \text{when } t < 0, \\
t & \text{when } 0 \leq t < 2, \\
6 - 2t & \text{when } 2 \leq t < 3, \\
0 & \text{when } t \geq 3
\end{cases}
\]

\[ x_1(t) = \]

\[ x_2(t) = \]
Now clearly $x_1(t)$ is the product of the signal $t$ with the following signal: Thus

$$x_1(t) = t[u(t) - u(t-2)].$$

Similarly, $x_2(t)$ is the product of the signal $6 - 2t$ with the following signal: Thus

$$x_2(t) = (6 - 2t)[u(t-2) - u(t-3)].$$

It follows that

$$x(t) = t[u(t) - u(t-2)] + (6 - 2t)[u(t-2) - u(t-3)].$$
1.1.7 Even and Odd Functions

Definition 1.1.29 A continuous-time signal $x(t)$ is an even function when
\[ x(-t) = x(t), \]
and is an odd function when
\[ x(-t) = -x(t). \]
Likewise, a discrete-time signal $x[k]$ is an even function when
\[ x[-k] = x[k], \]
and is an odd function when
\[ x[-k] = -x[k]. \]

See Fig. 1.13.

Remark 1.1.30 The signals
\[ x(t) = t^2 + 1 \quad \text{and} \quad x(t) = \cos(\omega t) \]
are clearly examples of even functions, while the signals
\[ x(t) = t \quad \text{and} \quad x(t) = \sin(\omega t) \]
are examples of odd functions, and the signals
\[ x(t) = 1 + t \quad \text{and} \quad x(t) = \cos(\omega t + \pi/4) \]
are neither even nor odd functions.
Remark 1.1.31 If $z(t)$ is a continuous-time odd signal then clearly we have $z(0) = 0$. Similarly if $z[k]$ is a discrete-time odd signal.

Remark 1.1.32 Every continuous-time signal $z(t)$ can be written as the sum of an even continuous-time signal and an odd continuous-time signal. Indeed, we can always put

$$z(t) = z_e(t) + z_o(t),$$

where

$$z_e(t) = \frac{1}{2} [z(t) + z(-t)], \quad z_o(t) = \frac{1}{2} [z(t) - z(-t)],$$

and it is trivially seen that $z_e(t)$ is an even function and $z_o(t)$ is an odd function. Likewise, every discrete-time signal $z[k]$ can be written as the sum of an even function and an odd function, namely

$$z[k] = z_e[k] + z_o[k],$$

where

$$z_e[k] = \frac{1}{2} [z[k] + z[-k]], \quad z_o[k] = \frac{1}{2} [z[k] - z[-k]],$$

are even and odd functions respectively.

Remark 1.1.33 For even functions $z(t)$ we have

$$\int_{-a}^{a} z(t) dt = 2 \int_{0}^{a} z(t) dt,$$

and for odd functions we have

$$\int_{-a}^{a} z(t) dt = 0.$$

These simplifying properties will be useful when we deal with Fourier series.
1.2 Systems

Having formulated the notion of signals in the preceding section, we come now to the other main idea of this course, namely a system:

Definition 1.2.1 A system is a device which transforms an input signal into an output signal, and the output signal is called the response of the system to the input signal. The system is called

\[ X(t) \xrightarrow{\text{input}} \text{SYSTEM} \xrightarrow{\text{output}} Y(t) \]

a continuous-time system when it transforms a continuous-time input signal into a continuous-time output signal, and is called a discrete-time system when it transforms a discrete-time input signal into a discrete-time output signal.

Remark 1.2.2 There is an important class of systems, called hybrid systems, which either transform a continuous-time signal into a discrete-time signal (e.g. A/D converters) or transform a discrete-time signal into a continuous-time signal (e.g. D/A converters). In this course we mainly study continuous-time systems and discrete-time systems.

Example 1.2.3 (a) The system is a resistor \( R \), the continuous-time input signal is the current \( i(t) \) through \( R \), and the continuous-time output signal is the voltage \( v(t) \) across \( R \):

\[ v(t) = Ri(t). \]

This is a continuous-time system.

(b) The system is a capacitor \( C \), the continuous-time input signal is the current \( i(t) \) through \( C \), and the continuous-time output signal is the voltage \( v(t) \) across \( C \):

\[ v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) \, d\tau. \]

This is a continuous-time system.
(c) The system is a "moving average filter" with discrete-time input signal $x[k]$ and discrete-time output signal $y[k]$ given by

$$y[k] = \frac{1}{3} (x[k] + x[k - 1] + x[k - 2]).$$

This is a discrete-time system.

The concept of a system is enormously broad. In order to be able to study systems we must take advantage of various simplifying properties which many systems have. These properties are

- Linearity
- Time invariance
- Causality.

We next formulate these properties precisely.
1.2.1 Linear Systems

As especially important class of systems has the following linearity or superposition property:

**Definition 1.2.4** A continuous-time system is linear when the following superposition property holds: If \( y_1(t) \) is the response of the system to the input signal \( x_1(t) \) and \( y_2(t) \) is the response of the system to the input signal \( x_2(t) \) then:

1. \( y_1(t) + y_2(t) \) is the response of the system to the input signal \( x_1(t) + x_2(t) \).
2. \( c y_1(t) \) is the response of the system to the input signal \( c x_1(t) \) for any complex constant \( c \).

The notion of linearity for discrete-time systems is formulated in exactly the same way.

**Example 1.2.5** Let \( x(t) \) be the input signal and \( y(t) \) be the output signal of a continuous-time system. Then the following systems are clearly linear:

\[
y(t) = 2x(t), \quad y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau.
\]

On the other hand, the following system **fails** to be linear:

\[
y(t) = x^2(t).
\]

One can just as easily come up with similar examples in the discrete-time case.

**Remark 1.2.6** Systems which fail to be linear are called **nonlinear** systems.

**Remark 1.2.7** Linear systems have the important property that the response to an input which is the zero signal (recall Remark 1.1.5) is likewise the zero signal. To see this consider the case of a continuous-time linear system, and suppose that \( y(t) \) is the response to the input \( x(t) \) which is the zero signal. Then, by linearity, \( 2y(t) \) is the response to the input signal \( 2x(t) \). But \( x(t) \) and \( 2x(t) \) are clearly identical (since \( x(t) \) is the zero signal), thus the corresponding responses \( y(t) \) and \( 2y(t) \) are also identical, namely

\[
y(t) = 2y(t) \quad \text{for all real } t.
\]

This shows that \( y(t) = 0 \) for all \( t \), as required. An identical argument works for discrete-time systems.
Example 1.2.8 Consider the continuous-time system for which the input $x(t)$ and output $y(t)$ are related by

$$y(t) = 1 + x(t).$$

Clearly this cannot be a linear system since the response to the zero input signal is the non-zero output signal $y(t) = 1$ for all $t$.

Example 1.2.9 A system is a series combination of a resistor $R$ and a capacitor $C$. The input signal is the current $i(t)$ through the series combination of $R$ and $C$, and the output signal is the voltage $v(t)$ across the series combination of $R$ and $C$. Clearly

$$v(t) = R i(t) + \frac{1}{C} \int_{-\infty}^{t} i(\tau) \, d\tau,$$

and the system is linear.
1.2.2 Time Invariance

Informally a system is time invariant when its properties do not change with time. We make this idea precise as follows:

Definition 1.2.10 A continuous-time system is time invariant when a time-shift in the input signal causes the same time-shift in the output signal. Specifically, if \( y_1(t) \) is the system response to an input signal \( x_1(t) \) then, for any real \( t_0 \), the response of the system to the input signal \( x_2(t) \) given by

\[
x_2(t) = x_1(t - t_0)
\]

is the signal \( y_2(t) \) given by

\[
y_2(t) = y_1(t - t_0).
\]

Likewise, a discrete-time system is time invariant when, if \( y_1[k] \) is the system response to an input signal \( x_1[k] \) then, for any integer \( k_0 \), the response of the system to the input signal \( x_2[k] \) given by

\[
x_2[k] = x_1[k - k_0]
\]

is the signal \( y_2[k] \) given by

\[
y_2[k] = y_1[k - k_0].
\]

Example 1.2.11 Consider the continuous-time nonlinear system given by

\[
y(t) = x^2(t).
\]

To check for time invariance let \( y_1(t) \) be the response to an input signal \( x_1(t) \), thus

\[
y_1(t) = x_1^2(t).
\]

Fix some \( t_0 \) and define the input signal

\[
x_2(t) = x_1(t - t_0).
\]

Then the corresponding output is the signal

\[
y_2(t) = x_1^2(t - t_0) = x_2^2(t - t_0)
\]

Moreover from (1.21) we have

\[
y_1(t - t_0) = x_1^2(t - t_0).
\]

Comparing (1.22) with (1.23) shows that \( y_2(t) = y_1(t - t_0) \), thus the system is time invariant.
Example 1.2.12 Consider the discrete-time system given by

\[ y[k] = kx[k]. \]

Let \( y_1[k] \) be the response to the input signal \( x_1[k] \) namely

\[ y[k] = kx[k]. \tag{1.24} \]

Fix some integer \( k_0 \) and let \( y_2[k] \) be the response to the input signal given by

\[ x_2[k] = x_1[k - k_0], \]

namely

\[ y_2[k] = kx_2[k] = kx_1[k - k_0]. \tag{1.25} \]

From (1.24) we have

\[ y_1[k - k_0] = (k - k_0)x_1[k - k_0] \neq kx_1[k - k_0], \]

thus from (1.25) we see \( y_1[k - k_0] \neq y_2[k] \). It follows that the system fails to be time invariant.

Remark 1.2.13 Systems which fail to be time invariant are called **time varying**.

Example 1.2.14 Consider the linear continuous-time system seen in Example 1.2.9 and let \( v_1(t) \) be the response to the input signal \( i_1(t) \), namely

\[ v_1(t) = Ri_1(t) + \frac{1}{C} \int_{-\infty}^{t} i_1(\tau) \, d\tau. \tag{1.26} \]

Fix some \( t_0 \) and let \( v_2(t) \) be the response to the input signal \( i_2(t) \) defined by

\[ i_2(t) = i_1(t - t_0), \]

thus

\[ v_2(t) = Ri_2(t) + \frac{1}{C} \int_{-\infty}^{t} i_2(\tau) \, d\tau \]

\[ = Ri_1(t - t_0) + \frac{1}{C} \int_{-\infty}^{t} i_1(\tau - t_0) \, d\tau. \tag{1.27} \]

Taking \( \tau = t - t_0 \) we see that

\[ \int_{-\infty}^{t} i_1(\tau - t_0) \, d\tau = \int_{-\infty}^{t-t_0} i_1(v) \, dv, \]

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and therefore from (1.27),

\[ v_2(t) = R_i(t - t_0) + \frac{1}{C} \int_{-\infty}^{t-t_0} i_1(r) dr. \]

Also, from (1.26) we must have

\[ v_1(t - t_0) = R_i(t - t_0) + \frac{1}{C} \int_{-\infty}^{t-t_0} i_1(r) dr. \]

Comparing (1.28) and (1.29) gives \( v_2(t) = v_1(t - t_0) \), showing that the system is time invariant.

**Remark 1.2.13** A useful consequence of linearity and time invariance is the following: suppose that \( y_u(t) \) is the response of a continuous-time linear time invariant system to a unit step input \( u(t) \), and we must find the response of the system to an input signal \( x(t) \) given by

\[ x(t) \triangleq \begin{cases} 
0 & \text{when } t < 0, \\
A & \text{when } 0 \leq t \leq T, \\
0 & \text{when } t \geq T.
\end{cases} \]

Clearly we have

\[ x(t) = A[u(t) - u(t - T)]. \]

See Fig. 1.14. By time invariance of the system we know that \( y_u(t - T) \) is the response to the input signal \( u(t - T) \), hence by linearity we know that \(-y_u(t - T)\) is the response to the input signal \(-u(t - T)\). Thus, again by linearity, we see that \( y_u(t) - y_u(t - T) \) is the response to the input signal \( u(t) - u(t - T) \). Again by linearity it follows that

\[ A[y_u(t) - y_u(t - T)] \]

is the response to the input signal \( x(t) \).
1.2.3 Causality

Informally a system is causal when it does not "look ahead" in time and anticipate values of the input signal after instant $t$ when deciding the value of the output signal at the instant $t$. This informal idea is precisely formulated as follows:

**Definition 1.2.16** A continuous-time system is causal under the following conditions: when $x_1(t)$ and $x_2(t)$ are input signals with corresponding outputs $y_1(t)$ and $y_2(t)$, and such that

$$x_1(t) = x_2(t), \quad \text{for all } t \leq t_0,$$

for a given instant $t_0$, it follows that

$$y_1(t_0) = y_2(t_0).$$

See Fig. 1.15. Likewise, a discrete-time system is causal under the following conditions: when $x_1[k]$ and $x_2[k]$ are input signals with corresponding outputs $y_1[k]$ and $y_2[k]$, and such that

$$x_1[k] = x_2[k], \quad \text{for all } k \leq k_0,$$

for a given instant $k_0$, it follows that

$$y_1[k_0] = y_2[k_0].$$

**Remark 1.2.17** Thus, causality in the continuous-time context means that whenever two input signals agree over their entire history up until some instant $t_0$, then the corresponding outputs agree at the same instant $t_0$ and hence of course also agree at all earlier instants $t \leq t_0$. This in turn means that the output of a causal system depends at instant $t_0$ only on the input signal $x(t)$ for $t \leq t_0$, and in no way depends on the input signal $x(t)$ for $t > t_0$. An exactly analogous interpretation is valid for the discrete-time case.
Example 1.2.18 Consider the RC system of Example 1.2.9 where the output signal (the voltage \( v(t) \)) is related to the input signal (the current \( i(t) \)) by

\[
v(t_0) = Ri(t) + \frac{1}{C} \int_{-\infty}^{t_0} i(\tau) \, d\tau.
\]

It is trivially clear that this is a causal system.

Example 1.2.19 Consider a discrete-time system with the output signal \( y[k] \) related to the input signal \( x[k] \) by

\[
y[k] = x[k] + x[k + 1] + x[k + 2] + x[k + 3] + x[k + 4].
\]

This is clearly not a causal system. Systems such as this which fail to be causal are called noncausal or anticipative systems.

Remark 1.2.20 Both causal and noncausal systems occur widely in practice. Systems which arise in the area of feedback control systems are usually causal, while on the other hand noncausal systems occur frequently in signal processing. In this course we shall encounter causal and noncausal systems, although the main emphasis will be on causal systems.

1.2.4 System Models: Electrical Networks

We have dealt with the notion of a system rather abstractly as a device which converts an input signal into an output signal. The question arises: how does one proceed from a given physical system, such as an electrical network, a mechanical system, or an economic system, to a mathematical relation which defines how the system transforms the input signal into the output signal? Such a relation is called a system model. Here we are going to derive system models for some electrical networks. The basic tools for deriving these models are the Kirchhoff current and voltage laws, and the terminal characteristics of devices such as resistors, capacitors, inductors, diodes, and transistors etc. In the following examples we shall derive system models in the form of differential equations:

Example 1.2.21 In the electrical network shown the input signal is the applied voltage \( x(t) \) and the output signal is the the current \( y(t) \). By KVL around the circuit:

\[
x(t) = v_1(t) + v_2(t) + v_3(t).
\]

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Now by the terminal relations for resistor, inductor and capacitor:

\[ V_1(t) = R_1 y(t), \]
\[ V_2(t) = L \frac{dy(t)}{dt}, \]
\[ V_3(t) = \frac{1}{C} \int_{-\infty}^{t} y(\tau) d\tau, \]

thus

\[ z(t) = R_1 y(t) + L \frac{dy(t)}{dt} + \frac{1}{C} \int_{-\infty}^{t} y(\tau) d\tau, \]

and upon taking derivatives:

\[ L \frac{d^2 y(t)}{dt^2} + R \frac{dy(t)}{dt} + \frac{1}{C} y(t) = \frac{dz(t)}{dt}. \]

In deriving the system model from a given physical system we often end up with system variables besides the input signal and output signal, and these variables must be eliminated to get a system model which involves only the input and output signals. A useful technique for doing this when dealing with differential equations is to introduce the D-operator by writing

\[ Dz(t) \triangleq \frac{dz(t)}{dt}. \]

for a given signal \( z(t) \), and treating \( D \) as if it were a number. An example best illustrates the idea:

**Example 1.2.22** In the network shown the input signal is the applied voltage \( z(t) \) and the output signal is the current \( y(t) \) through the 2-ohm resistor. A system model relating \( z(t) \) with \( y(t) \) must
be derived. Introduce mesh currents $y_1(t)$ and $y_2(t)$ as shown. It is apparent that

$$y(t) = y_1(t).$$

By KVL around loop for $y_1$:

$$z(t) = 2y_1(t) + 2Dy_1(t) + 3(y_1(t) - y_2(t)),$$
thus

(1.30) \hspace{1cm} (2D + 5)y_1(t) - 3y_2(t) = z(t).

By KVL around loop for $y_2$:

$$4y_2(t) + 3(y_2(t) - y_1(t)) + Dy_2(t) = 0,$$
thus

(1.31) \hspace{1cm} -3y_1(t) + (D + 7)y_2(t) = 0.

The input signal is $x(t)$ and the output signal is $y(t) = y_1(t)$. To relate $x(t)$ to $y(t)$ we must eliminate the signal $y_2(t)$ from (1.30) and (1.31) by algebraically manipulating these equations, a somewhat lengthy and error-prone method. In fact it is more efficient to proceed exactly as if $D$ were a number and write (1.30) and (1.31) in matrix form:

$$\begin{bmatrix} 2D + 5 & -3 \\ -3 & (D + 7) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \end{bmatrix}.$$

Again, regarding $D$ as a number, we can use Cramer’s rule to get $y_1(t)$ in terms of $x(t)$:

$$y_1(t) = \frac{\begin{vmatrix} x(t) & -3 \\ 0 & (D + 7) \end{vmatrix}}{\begin{vmatrix} 2D + 5 & -3 \\ -3 & (D + 7) \end{vmatrix}}.$$
thus

\[ y_1(t) = \frac{(D + T)x(t)}{2D^2 + 19D + 26}, \]

or

\[(2D^2 + 19D + 26)y_1(t) = (D + T)x(t).\]

Since \( y_1(t) = y(t) \) this can be written as

\[2\frac{d^2y(t)}{dt^2} + 19\frac{dy(t)}{dt} + 26y(t) = \frac{dx(t)}{dt} + 7x(t).\]