ECE-342 Problem Set 1: Introduction to Signals and Systems

1. Find the energy in the following signals:

(a) $x(t) = 2[u(t-1) - u(t-2)]$

(b) $x(t) = t[u(t) - u(t-5)]$

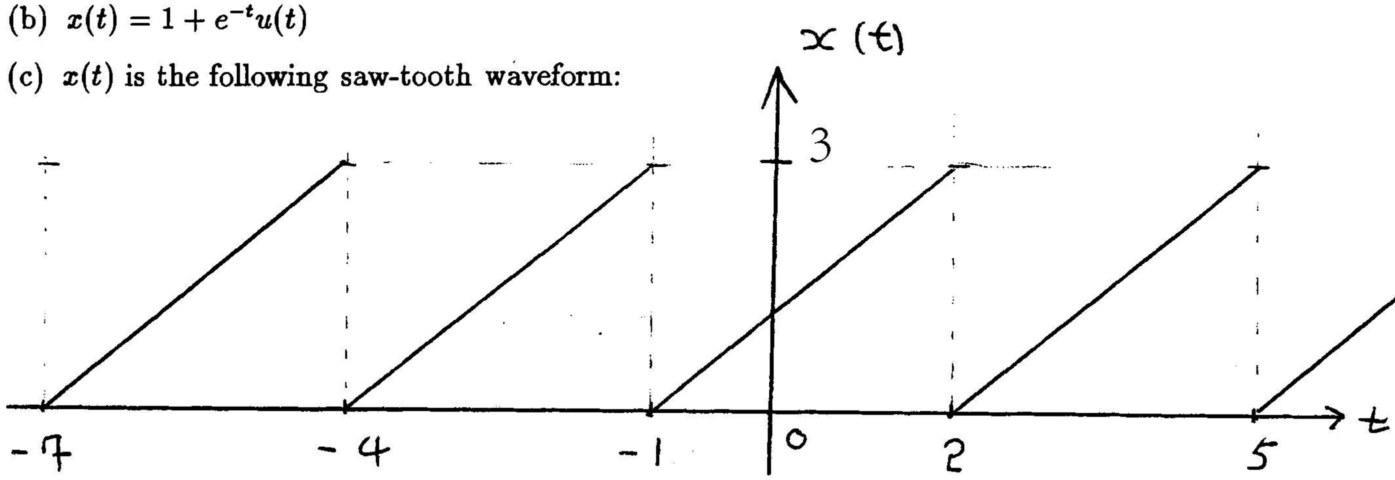
(c) $x[k] = \left(\frac{1}{2}\right)^k u[k-3]$.

2. Find the average power in the following signals:

(a) $x(t) = u(-t)$

(b) $x(t) = 1 + e^{-t}u(t)$

(c) $x(t)$ is the following saw-tooth waveform:



3. Are the following signals periodic?

(a) $x(t) = \cos(t/2) + \sin(3t/5)$

(b) $x(t) = \cos^2(t/2) + \sin^2(3t/5)$

(c) $x(t) = t + \cos(3t/4)$

In the case of the periodic signals determine a period.

4. Determine the even and odd parts of the following signals:

(a) $x(t) = \cos(t) + \sin(t)$

(b) $x(t) = e^t$

(c) $x(t) = e^t u(t)$

(d) $x(t) = \sin(t)u(t)$

(e) $x(t) = e^{j\omega t}$.

5. Determine which of the following systems are linear:

(a) $\dot{y}(t) + y(t) = \cos(x(t)), \quad y(0) = 0$

(b) $\dot{y}(t) + e^{-t}y(t) = x(t), \quad y(0) = 1$

(c) $\dot{y}(t) + 10y(t) + 5 = x(t), \quad y(0) = 0.$

6. Determine if the following system is time-invariant.

$$y(t) = \int_{-\infty}^{t} \tau x(\tau) d\tau.$$

7. Let $y_0(t)$ be the response of a linear time invariant system to an input pulse $x_0(t)$ of the following form:

$$x_0(t) = \begin{cases} 0 & \text{for all } t \leq 0, \\ 1 & \text{for all } 0 < t \leq 1, \\ 0 & \text{for all } 1 < t. \end{cases}$$

Now suppose the following input is applied:

$$x(t) = \begin{cases} 0 & \text{for all } t \leq 0, \\ 1 & \text{for all } 0 < t \leq 1, \\ 3 & \text{for all } 1 < t \leq 2, \\ 0 & \text{for all } 2 < t \leq 3, \\ -2 & \text{for all } 3 < t \leq 4, \\ 0 & \text{for all } 4 < t, \end{cases}$$

Determine the output in terms of shifted versions of $y_0(t)$.

8. A linear system has the following properties:

(a) its response to the step input $u(t)$ is the signal $y(t) = t$.

(b) its response to the delayed step input is the signal $y(t) = t/2$.

Is the system

(i) causal

(ii) time invariant

Determine the response of this system to the input signal

$$x(t) = \begin{cases} 0 & \text{for all } t \leq 0, \\ 2 & \text{for all } 0 < t \leq 1, \\ 1 & \text{for all } 1 < t. \end{cases}$$

9. In a system the input signal $x(t)$ is related to the output signal $y(t)$ by

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau, \qquad \text{for all } t,$$

where the function $h(t)$ is given by

$$h(t) = \begin{cases} 0 & \text{for all } t < -1, \\ 1 & \text{for all } -1 \leq t < 0, \\ 0 & \text{for all } 0 \leq t. \end{cases}$$

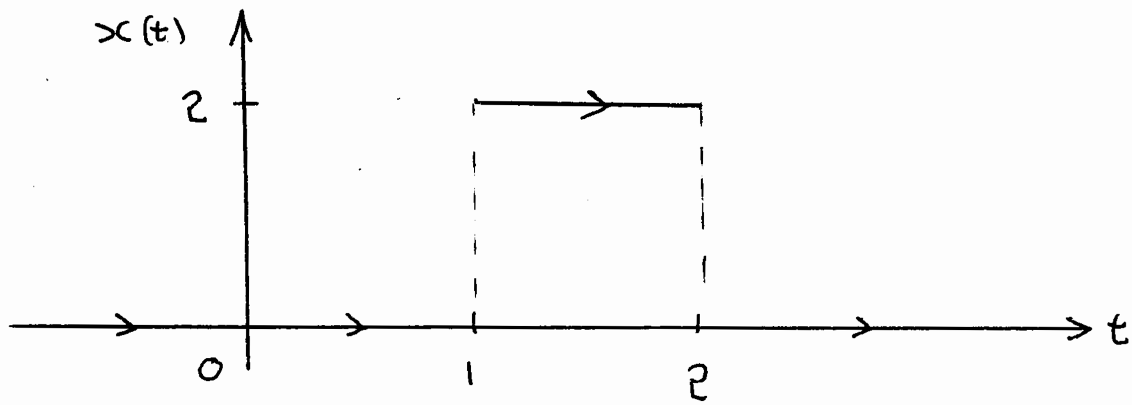
Determine if the system is causal or not.

Problem Set 1

1. (a) $\quad x(t) = 2\left[u(t-1) - u(t-2)\right]$

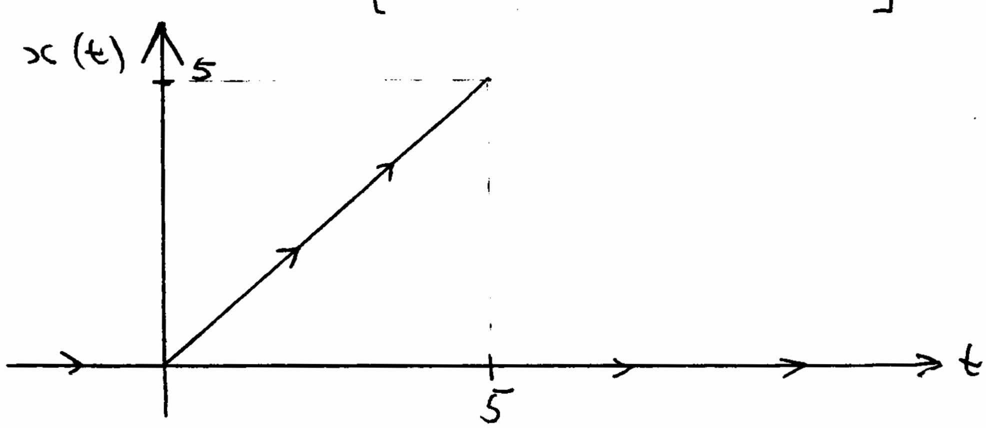
Energy in signal:

$$E = \int_{-\infty}^{\infty} x^2(\tau)\, d\tau$$



Thus $\quad E = \int_1^2 4 \cdot d\tau = 4.$

1. (b) $\quad x(t) = t\left[u(t) - u(t-5)\right]$



Energy in signal:

$$E = \int_{-\infty}^{\infty} x^2(\tau)\, d\tau$$

$$= \int_0^5 \tau^2 . d\tau$$

$$= \frac{\tau^3}{3} \Big|_0^5$$

$$= \frac{125}{3}.$$

1.(c) $x[k] = \left(\frac{1}{2}\right)^k . u[k-3]$

Energy in signal

$$E = \sum_{k=-\infty}^{\infty} x^2[k]$$

Now $u[k-3] = 1$ iff $k \geq 3$.

Thus

$$E = \sum_{k \geq 3} \left(\frac{1}{2}\right)^{2k}$$

$$= \sum_{k \geq 3} \left(\frac{1}{4}\right)^{k}$$

$$= \left(\frac{1}{4}\right)^{3} \sum_{k \geq 0} \left(\frac{1}{4}\right)^{k}$$

$$= \left(\frac{1}{4}\right)^{3} \frac{1}{1 - 1/4}$$

$$= \left(\frac{4}{3}\right) \left(\frac{1}{4}\right)^{3}$$

$$= \frac{1}{3\ (4)^2}.$$

2. (a) $x(t) = u(-t)$

average power is

$$\underline{P} = \lim_{a \to \infty} \underline{P}(a)$$

where

$$\underline{P}(a) = \frac{1}{2a} \int_{-a}^{a} x^2(\tau)\, d\tau$$

Now $x(t) = \begin{cases} 1 & \text{for all } t \leq 0 \\ 0 & \text{" } \quad t > 0 \end{cases}$

thus

$$\underline{P}(a) = \frac{1}{2a} \int_{-a}^{0} d\tau$$

$$= \frac{1}{2}.$$

thus $\underline{P} = \frac{1}{2}.$

2 (b) $x(t) = 1 + e^{-t} . u(t)$

Put

$$P(a) = \frac{1}{2a} \int_{-a}^{a} x^2(\tau) . d\tau$$

$$= \frac{1}{2a} \int_{-a}^{a} [1 + e^{-\tau} . u(\tau)]^2 d\tau$$

$$= \frac{1}{2a} \int_{-a}^{a} [1 + 2e^{-\tau} . u(\tau) + e^{-2\tau} u(\tau)] d$$

(since $u(\tau) = u^2(\tau)$)

$$= 1 + \frac{1}{2a} \int_{0}^{a} 2e^{-\tau} . d\tau$$

$$+ \frac{1}{2a} \int_{0}^{a} e^{-2\tau} . d\tau$$

$$= 1 + \frac{1}{2a} \left\{ -2e^{-\tau} \Big|_0^a + \left[-\frac{e^{-2\tau}}{2} \right]_0^a \right\}$$

$$= 1 + \frac{1}{2a}\left\{\left(1 - 2e^{-a}\right) + \left(\frac{1}{2} - \frac{1}{2}e^{-2a}\right)\right\}$$

$$= 1 + \left\{\frac{1}{2a}\left(\frac{3}{2}\right) - \frac{2e^{-a}}{2a} - \frac{e^{-2a}}{4a}\right\}$$

Now clearly

$$\lim_{a\to\infty}\left\{\frac{1}{2a}\left(\frac{3}{2}\right) - \frac{2e^{-a}}{2a} - \frac{e^{-2a}}{4a}\right\}$$

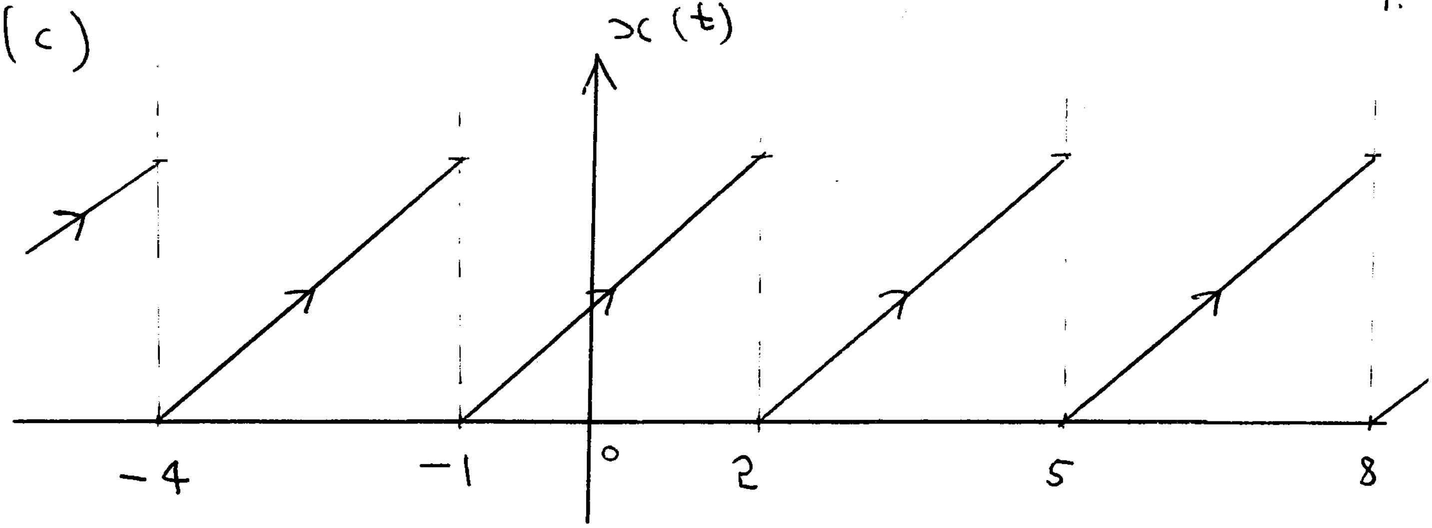
$$= 0$$

hence

$$\underline{P} = \lim_{a\to\infty} \underline{P}(a)$$

$$= 1.$$

2 (c)



The signal $x(t)$ is periodic with a period

$$T = 3.$$

Then average power is

$$P = \frac{1}{T}\int_0^T x^2(t)\,dt$$

$$= \frac{1}{3}\int_0^3 t^2 \cdot dt$$

$$= 3.$$

3 (a) $x(t) = x_1(t) + x_2(t)$

for

$x_1(t) \triangleq \cos(t/2)$ $\qquad x_3(t) \triangleq \sin(3t/5)$.

Thus

$$x_1(t) = \cos(\omega_1 t), \quad \omega_1 \triangleq 1/2,$$

i.e. $x_1(t)$ is periodic with period

$$T_1 = \frac{2\pi}{\omega_1} = 4\pi.$$

Also

$$x_2(t) \triangleq \sin(3t/5)$$

$$= \cos\left(\frac{3t}{5} - \frac{\pi}{2}\right)$$

$$= \cos(\omega_2 t - \pi/2)$$

for $\omega_2 \triangleq \dfrac{3}{5}$

i.e. $x_2(t)$ is periodic with period

$$T_2 = \frac{2\pi}{\omega_2} = \frac{10\pi}{3}.$$

Then

$$\frac{T_1}{T_2} = \frac{6}{5}.$$

which is rational.

Put

$$T \triangleq 5T_1 = 6T_2$$
$$= 20\pi$$

Then it follows exactly as in Fact 1.1.10 that $x(t)$ is periodic with a period $T = 20\pi$.

3 (b) $x(t) \triangleq x_1(t) + x_2(t)$

for

$$x_1(t) \triangleq \cos^2(t/2)$$

$$x_2(t) \triangleq \sin^2(3t/5)$$

Since $\cos(t/2)$ has been seen to be periodic with period $T_1 = 4\pi$, (see 3(a))

it follows that $\cos^2(t/2)$ is also periodic with period with $T_1 = 4\pi$

i.e. $x_1(t)$ is periodic with period

$$T_1 = 4\pi.$$

Similarly $x_2(t)$ is periodic with a period $T_2 = 10\pi/3$.

Therefore, as in 3(a) we see that

$x(t)$ is periodic with period

$$T = 20\pi.$$

3 (c) $x(t) = t + \cos(3t/4)$

Clearly

$x(t) > 0$ for all $t \geq 1$

and

$x(t) < 0$ for all $t < -1$.

Thus $x(t)$ cannot possibly be periodic.

4. (a) $x(t) = \cos(t) + \sin(t)$.

even part of $x(t)$:

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$= \frac{\cos(t) + \cos(-t) + \sin(t) + \sin(-t)}{2}$$

Now $\cos(t) = \cos(-t)$

and $\sin(t) = -\sin(-t)$ — ①

hence

$$x_e(t) = \cos(t).$$

odd part of $x(t)$:

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$= \frac{\cos(t) + \sin(t) - [\cos(-t) + \sin(-t)]}{2}$$

$$= \sin(t) \qquad \text{(see ①)}.$$

4. (b) $x(t) = e^{t}$.

even part of $x(t)$:

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$= \frac{e^{t} + e^{-t}}{2}$$

$$= \cosh(t)$$

odd part of $x(t)$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$= \frac{e^{t} - e^{-t}}{2}$$

$$= \sinh(t).$$

4 (c) $x(t) = e^{t} \cdot u(t)$

even part

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$= \frac{1}{2}\left[u(t)e^{t} + u(-t)\,e^{-t}\right]$$

odd part:

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$= \frac{1}{2}\left[u(t)e^{t} - u(-t)\,e^{-t}\right]$$

4 (d) $x(t) = \sin(t) \cdot u(t)$

even part

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$= \frac{\sin(t) \cdot u(t) + \sin(-t) \cdot u(-t)}{2}$$

$$= \frac{\sin(t) \cdot u(t) - \sin(t) \cdot u(-t)}{2}$$

$$\therefore \quad x_e(t) = \frac{1}{2} \cdot \sin(t)\left[u(t) - u(-t)\right]$$

$$= \begin{cases} \frac{1}{2}\sin(t) & t \geq 0 \\ -\frac{1}{2}\sin(t) & t \leq 0. \end{cases}$$

odd part

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$= \frac{\sin(t) \cdot u(t) - \sin(-t) \cdot u(-t)}{2}$$

$$= \frac{\sin(t) \cdot u(t) + \sin(t) \cdot u(-t)}{2}$$

$$= \frac{1}{2} \cdot \sin(t)\left[u(t) + u(-t)\right]$$

$$= \frac{1}{2} \sin(t).$$

4 (e) $x(t) = e^{j\omega t}$

$$= \cos(\omega t) + j \sin(\omega t).$$

even part:

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$= \frac{1}{2}\left[\cos(\omega t) + j\sin(\omega t) + \cos(-\omega t) + j\sin(-\omega t)\right]$$

$$= \cos(\omega t)$$

odd part

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$= j\sin(\omega t).$$

5(a) $$\begin{cases} \dot{y}(t) + y(t) = \cos(x(t)) \\ y(0) = 0. \end{cases}$$

We guess that system is <u>nonlinear</u> because of the nonlinear cosine function on RHS, but how can we convincingly show this?

Here is one approach to this:

Fix an input signal $x_1(t)$ given by

$$x_1(t) = \alpha \qquad \text{all } t,$$

for some constant $\alpha \neq 0$.

Let $y_1(t)$ be response to input $x_1(t)$.

Thus

$$\text{①} \quad \begin{cases} \dot{y}_1(t) + y_1(t) = \beta \\ y_1(0) = 0 \end{cases}$$

where

$$\beta \triangleq \cos(\alpha)$$

Now define input signal $x_2(t)$ by

(2) $$x_2(t) \triangleq -x_1(t), \quad \text{all } t,$$

thus
$$x_2(t) = -\alpha, \quad \text{all } t,$$

cosine is even

thus

(3) $$\cos(x_2(t)) = \cos(-\alpha) = \cos(\alpha) = \beta$$

for all t.

Let $y_2(t)$ be response to input $x_2(t)$.

Then
$$\begin{cases} \dot{y}_2(t) + y_2(t) = \cos(x_2(t)) \\ y_2(0) = 0. \end{cases}$$

ie from (3)

(4) $$\begin{cases} \dot{y}_2(t) + y_2(t) = \beta \\ y_2(0) = 0. \end{cases}$$

Comparing (1) and (4) we see that the differential equations for $y_1(t)$ and $y_2(t)$ are <u>identical</u>, with <u>identical</u> initial conditions, so we must have

(5) $$y_1(t) = y_2(t), \quad \text{all } t$$

Now, if the given system were linear, then, from (2), we would necessarily have

$$y_2(t) = -y_1(t), \quad \text{all } t,$$

(by scaling property of linearity) which contradicts what we have shown in (5).

Thus system <u>cannot</u> be linear, thus is <u>nonlinear.</u>

5 (b)
$$\begin{cases} \dot{y}(t) + e^{-t} y(t) = x(t) \\ y(0) = 1. \end{cases}$$

If system is linear then zero input signal $x(t)$ **must** result in a zero output signal $y(t)$, namely $y(t) \equiv 0$, all t.

But this contradicts the fact that

$$y(0) = 1.$$

$\therefore$ system is **nonlinear**.

5 (c)
$$\left.\begin{aligned} \dot{y}(t) + 10\,y(t) + 5 &= x(t) \\ y(0) &= 0. \end{aligned}\right\} \quad (1)$$

Let $y_1(t)$ be response to the input $x_1(t)$

" $y_2(t)$ " " " " $x_2(t)$.

Then

$$\left.\begin{aligned} \dot{y}_i(t) + 10\,y_i(t) + 5 &= x_i(t) \\ y_i(0) &= 0 \end{aligned}\right\} \quad (2)$$

for $i = 1, 2$.

Put

$$\left.\begin{aligned} x(t) &\triangleq x_1(t) + x_2(t) \\ y(t) &\triangleq y_1(t) + y_2(t) \end{aligned}\right\} \quad (3)$$

adding (2) for $i = 1, 2$ and using (3) gives

$$\left.\begin{aligned} \dot{y}(t) + 10\,y(t) + 10 &= x(t) \\ y(0) &= 0 \end{aligned}\right\} \quad (4)$$

Now, if given system (1) were linear, then $y(t)$ is the response to the input $x(t)$ (see (3)), so that

$$\left.\begin{aligned} \dot{y}(t) + 10\,y(t) + 5 &= x(t) \\ y(0) &= 0 \end{aligned}\right\} \quad (5)$$

which is obviously different from (4).

Thus the system is <u>nonlinear</u>.

6. (a) $$y(t) = \int_{-\infty}^{t} \tau . x(\tau) . d\tau$$

Let $x_1(t)$ be input with corresponding output $y_1(t)$. also for some fixed instant $t_0 \neq 0$ put

$$x_2(t) \triangleq x_1(t - t_0)$$

and let $y_2(t)$ be the corresponding output namely

$$y_2(t) = \int_{-\infty}^{t} \tau . \overbrace{x_1(\tau - t_0)}^{x_2(\tau)} . d\tau$$

$$= \int_{-\infty}^{t-t_0} (\sigma + t_0) x_1(\sigma) d\sigma$$

(with $\sigma \triangleq \tau - t_0$)

$$\therefore \quad y_2(t) = \int_{-\infty}^{t-t_0} \sigma . x_1(\sigma) . d\sigma$$

$$+ t_0 \int_{-\infty}^{t-t_0} x_1(\sigma) d\sigma \qquad (1)$$

also

$$y_1(t) = \int_{-\infty}^{t} \tau\, x_1(\tau)\, d\tau$$

thus

$$y_1(t - t_0) = \int_{-\infty}^{t-t_0} \sigma . x_1(\sigma)\, d\sigma \quad \text{—— } ②$$

combining ① and ②

$$y_2(t) = y_1(t - t_0) + t_0 \int_{-\infty}^{t-t_0} x_1(\sigma)\, d\sigma .$$

③

Take e.g.

$$x_1(\sigma) = e^{\sigma}.$$

Then ③ gives

$$y_2(t) = y_1(t - t_0) + t_0 \int_{-\infty}^{t-t_0} e^{\sigma} . d\sigma$$

④

If system were time invariant then it would follow that

$$y_2(t) = y_1(t - t_0)$$

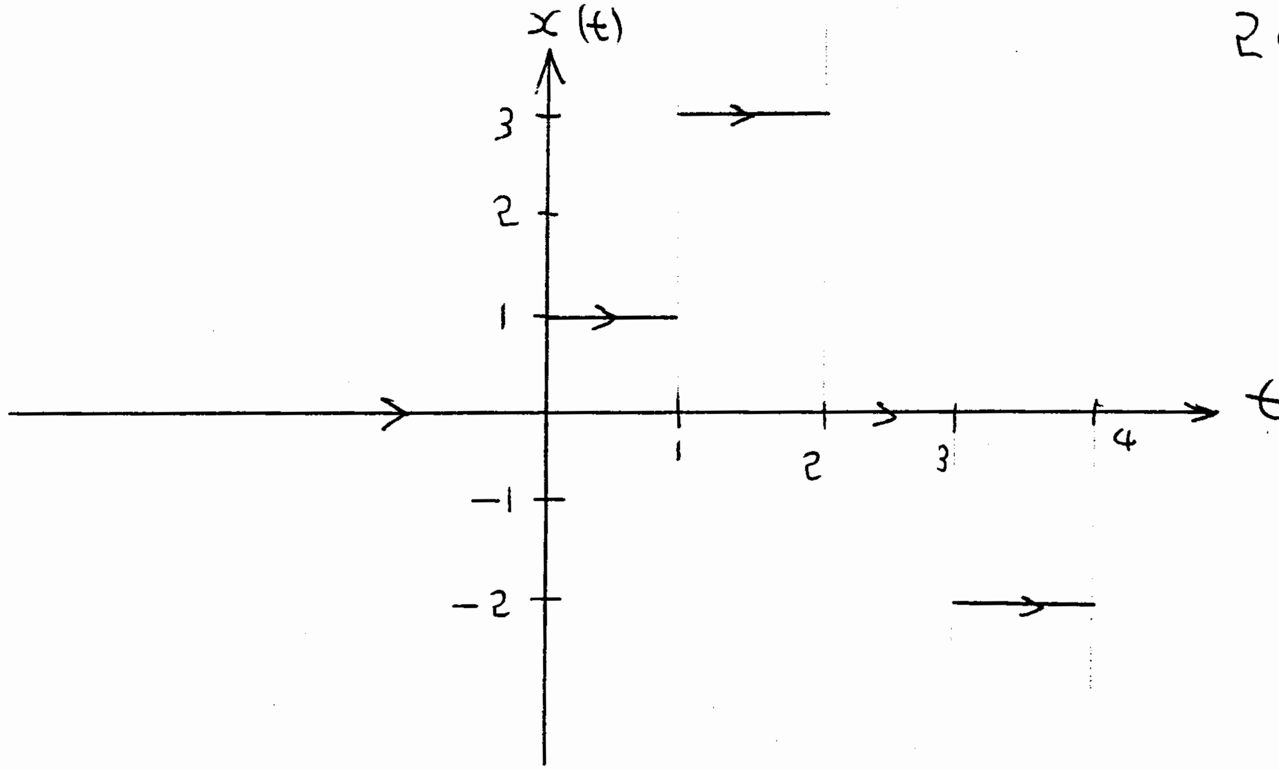
and combining with (4) gives

$$t_0 \int_{-\infty}^{t - t_0} e^{\sigma} . \, d\sigma = 0$$

which is impossible.

i.e. system is <u>time - varying.</u>

7.



Clearly

$$x(t) = x_0(t) + 3x_0(t-1) - 2x_0(t-3) \quad \text{—} (1)$$

System is time-invariant thus

response to input signal $x_0(t-1)$ is $y_0(t-1)$

" " " $x_0(t-3)$ " $y_0(t-3)$

∴ by linearity and (1) we see that response to input signal $x(t)$ is

$$y(t) = y_0(t) + 3y_0(t-1) - 2y_0(t-3).$$

8 Put $x_1(t) \triangleq u(t)$
$x_2(t) \triangleq u(t-1)$ } — ①

and let $y_1(t) \triangleq t$
$y_2(t) \triangleq \frac{t}{2}$ } — ②

be respective output signals.

(i) For causality observe that

$$x_1(t) = x_2(t) \quad \forall\, t \leq 0.$$

If system were causal then this would imply that

$$y_1(t) = y_2(t) \quad \text{for all } t \leq 0$$

which is clearly not the case. (from ②)

$\therefore$ system not causal.

(ii) if time invariance held then we must get

$$y_2(t) = y_1(t-1) \quad \text{—— ③}$$

(since we have

$$x_2(t) = x_1(t-1).$$

But (3) contradicts (2).

$\therefore$ system is not time invariant.

System input is

$$x(t) = 2u(t) - u(t-1)$$

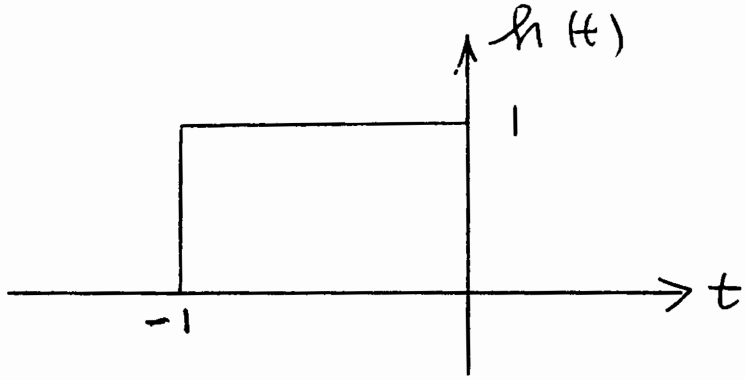
By linearity the corresponding output is

$$y(t) = 2y_1(t) - y_2(t)$$

$$= 2t - t/2$$

$$= \frac{3t}{2}.$$

9.



$$y(t) = \int_{-\infty}^{\infty} h(t-\tau) . x(\tau) . d\tau \text{ ——— } ①$$

Is the system causal?

Fix input signals

$$\left.\begin{array}{ll} x_1(t) = 0 & \text{for all } t \\ x_2(t) = h(t) & \text{" "} \end{array}\right\} \text{——— } ②$$

then $\quad x_1(t) = x_2(t) \quad$ for all $t \leq -1$

thus, if system is causal, then

$$y_1(-1) = y_2(-1)$$

$$\therefore \; y_2(-1) = 0 \text{ ——— } ③$$

(since ① and ② give $y_1(-1) = 0$).

Now $\quad y_2(-1) = \int_{-\infty}^{\infty} h(\tau)\, h(-1-\tau)\, d\tau$

$$\therefore \quad y_2(-1) = \int_{-1}^{0} h(-1-\tau)\, d\tau \quad \text{———} \quad (4)$$

$$\left(\text{since} \quad h(\tau) = \begin{cases} 1 & , \ -1 \leq \tau \leq 0 \\ 0 & , \ \text{otherwise} \end{cases} \right.$$

Now

$$-1 \leq -1-\tau \leq 0 \quad \text{iff} \quad -1 \leq \tau \leq 0$$

thus

$$h(-1-\tau) = 1 \quad \text{for all} \quad -1 \leq \tau \leq 0,$$

thus (4) gives

$$y_2(-1) = \int_{-1}^{0} 1 \, d\tau$$

$$= 1$$

which contradicts (3)

$\therefore$ system <u>cannot</u> be causal.