ECE-342 Problem Set 5:Periodic Inputs and Fourier Series

1. Determine the exponential Fourier coefficients for the signal

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \pi/4).$$

2. The input to an LTI system with impulse response

$$h(t) = e^{-t}u(t)$$

is the periodic signal

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t}$$

for

$$\omega_0 = 2\pi \quad \text{rads/sec.}$$

and

$$\begin{aligned} a_0 &= 1, \\ a_1 &= a_{-1} = \frac{1}{4}, \\ a_2 &= a_{-2} = \frac{1}{2}, \\ a_3 &= a_{-3} = \frac{1}{3}. \end{aligned}$$

Determine the exponential Fourier coefficients of the output signal.

3. A signal $x(t)$ is periodic with period T such that

$$x(t) = \frac{At}{T}, \qquad \text{for all } 0 \leq t < T.$$

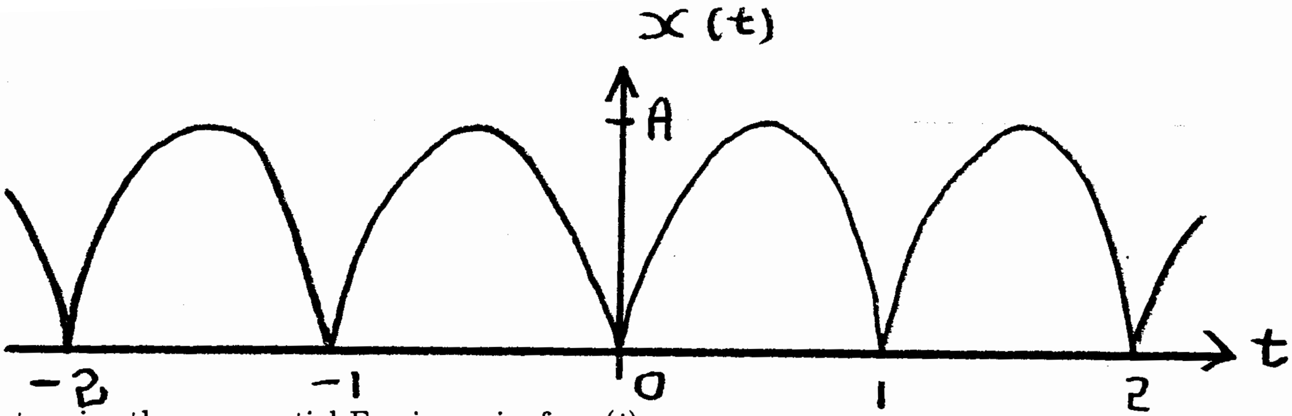
(a) Determine the exponential Fourier expansion for $x(t)$.

(b) Write a Fourier expansion for $x(t)$ in terms of the trigonometric functions $\cos(x)$ and $\sin(x)$.

4. A periodic signal with period $T = 1$ is given by

$$x(t) = A\sin(\pi t), \qquad \text{for all } 0 \leq t \leq 1,$$

as dipicted in the next figure.



(a) Determine the exponential Fourier series for $x(t)$.

(b) Write a Fourier expansion for $x(t)$ in terms of the trigonometric functions $\cos(x)$ and $\sin(x)$.

(c) The signal $x(t)$ is the input to the initially-at-rest system

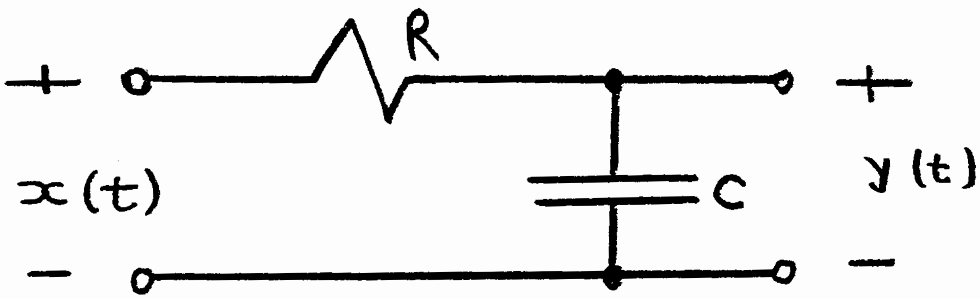
$$(D+1)y(t) = x(t).$$

Determine the exponential Fourier coefficients for the output signal.

5. The input signal to the RC circuit shown is the voltage

$$x(t) = 100\cos(t) + 10\cos(3t) + \cos(5t),$$

and the output signal is the voltage $y(t)$.



Determine output signal as a series in terms of the trigonometric functions $\cos(x)$ and $\sin(x)$.

6. Determine the average power in the signal

$$x(t) = \cos^2(20\pi t)\sin(10\pi t).$$

1. $x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t)$

$$+ \cos\left[2\omega_0 t + \pi/4\right]$$

We must determine exponential Fourier coefficients for $x(t)$. We can do this directly by evaluating

$$a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t}\, dt$$

But in this case it is easier to use Eulers' formulae as follows:

$$x(t) = 1 + \frac{[e^{j\omega_0 t} - e^{-j\omega_0 t}]}{2j}$$

$$+ 2\,\frac{[e^{j\omega_0 t} + e^{-j\omega_0 t}]}{2}$$

$$+ \frac{[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}]}{2}$$

Collecting terms gives

$$x(t) = 1 + \left[1 + \frac{1}{2j}\right]e^{j\omega_0 t}$$

$$+ \left[1 - \frac{1}{2j}\right]e^{-j\omega_0 t}$$

$$+ \frac{1}{2}e^{j\pi/4} \cdot e^{j2\omega_0 t} + \frac{1}{2}e^{-j\pi/4}e^{-j2\omega_0 t}$$

The exponential Fourier coefficients are thus

$$a_0 = 1$$

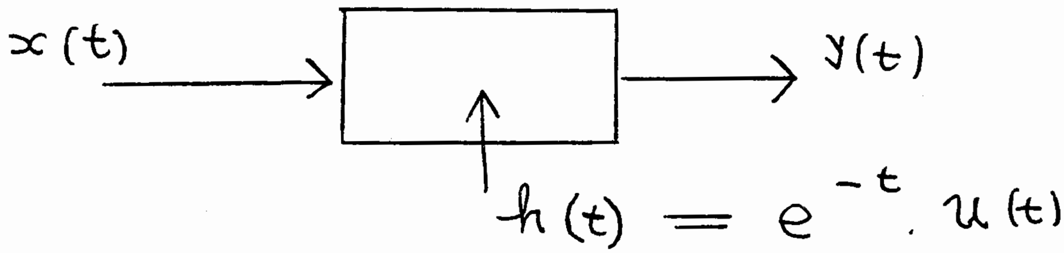
$$a_1 = 1 + \frac{1}{2j} = 1 - \frac{j}{2}$$

$$a_{-1} = 1 - \frac{1}{2j} = 1 + \frac{j}{2}$$

$$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1+j)$$

$$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1-j).$$

2.



Input signal is the periodic signal

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t} \quad \text{———} \quad (1)$$

for $\omega_0 = 2\pi$ rads/sec. ——— (2)

and

$$a_0 = 1$$

$$a_1 = a_{-1} = \frac{1}{4}$$

$$a_2 = a_{-2} = \frac{1}{2}$$

$$a_3 = a_{-3} = \frac{1}{3}$$

Now transfer function of the system is

$$H(s) \triangleq \mathcal{L}\{h(t)\}(s)$$

hence

$$H(s) = \frac{1}{s+1} \quad \text{———} \quad (3)$$

Now response to the input signal $e^{jk\omega_0 t}$

is

$$H(jk\omega_0)\, e^{jk\omega_0 t}.$$

Thus, by superposition, response to the input signal (1) is

$$y(t) = \sum_{k=-3}^{3} a_k H(jk\omega_0)\, e^{jk\omega_0 t}$$

Put

$$b_k \triangleq a_k H(jk\omega_0) \quad \text{———} \quad (4)$$

These are the exponential Fourier coefficients of the output signal $y(t)$.

From (3) we get

$$b_0 = a_0 H(0)$$

$$= a_0 = 1.$$

$$b_1 = a_1 H(j\omega_0)$$

$$= \frac{1}{4} \cdot H(j2\pi) = \frac{1}{4(1+j2\pi)}$$

$$b_{-1} = a_{-1} H(-j\omega_0)$$

$$= \frac{1}{4} H(-j2\pi) = \frac{1}{4(1-j2\pi)}$$

$$b_2 = a_2 H(2j\omega_0)$$

$$= \frac{1}{2} \cdot H(j4\pi) = \frac{1}{2(1+j4\pi)}$$

$$b_{-2} = a_{-2} H(-2j\omega_0)$$

$$= \frac{1}{2} \cdot H(-j4\pi) = \frac{1}{2(1-j4\pi)}$$

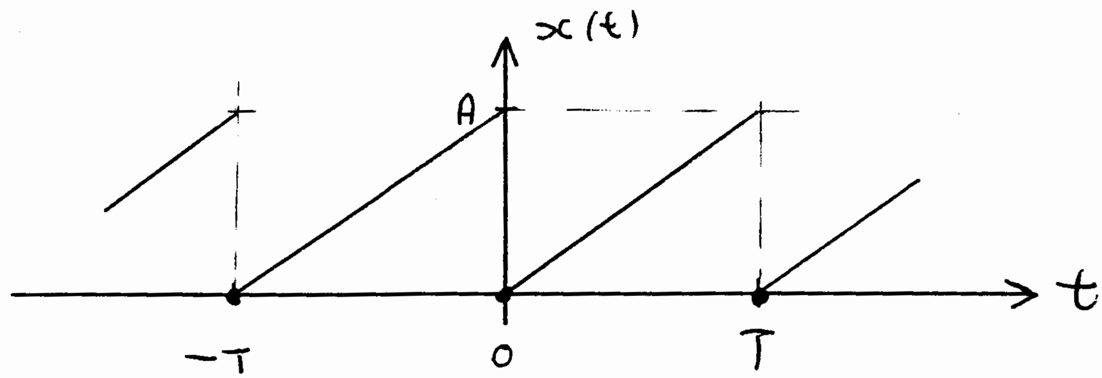
$$b_3 = a_3 H(3j\omega_0)$$

$$= \frac{1}{3} \cdot H(j6\pi) = \frac{1}{3(1+j6\pi)}$$

$$b_{-3} = a_{-3} H(-3j\omega_0) = \frac{1}{3(1-j6\pi)}$$

3. (a) $x(t) = \dfrac{At}{T}, \quad 0 \leq t < T$

$x(t) = x(t+T)$.



Exponential Fourier series coefficients are

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_o t}\, dt$$

where $\omega_o = \dfrac{2\pi}{T}$.

i.e.

$$a_k = \frac{1}{T}\cdot\int_0^T \frac{A}{T}t \cdot e^{-jk\omega_o t}\, dt$$

$$= \frac{A}{T^2}\int_0^T t\, e^{-jk\omega_o t}\, dt \quad \text{\_\_\_ (1)}$$

Now consider $k \neq 0$.

using integration - by - parts :

$$\int_0^T t e^{-jk\omega_0 t}\, dt = \int_0^T t \cdot \frac{d}{dt}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right] dt$$

$$= \int_0^T t \cdot d\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right)$$

$$= \frac{t \cdot e^{-jk\omega_0 t}}{-jk\omega_0}\Bigg|_0^T$$

$$- \int_0^T \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \cdot dt$$

$$= \frac{T e^{-jk\omega_0 T}}{-jk\omega_0} + \frac{1}{jk\omega_0}\int_0^T e^{-jk\omega_0 t} \cdot dt$$

$$= \frac{T \cdot e^{-jk2\pi}}{-jk\omega_0} + \frac{0}{jk\omega_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^T$$

where we have used the fact

$$\int_0^T e^{-jk\omega_0 t}\, dt = 0$$

when $k \neq 0$.

since

$$e^{-jk2\pi} = 1 \quad \text{for all } k$$

we get

$$\int_0^T t\, e^{-jk\omega_0 t}\, dt = \frac{T}{-jk\omega_0}$$

$$= \frac{jT}{k\omega_0} \quad \text{—— (2)}$$

From (1), (2), when $k \neq 0$ we have

$$a_k = j \cdot \frac{A}{T^2} \cdot \frac{T}{k\omega_0}$$

$$= j \cdot \frac{A}{Tk\omega_0}$$

$$= j\,\frac{A}{2\pi k}$$

also,

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{A}{T^2}\int_0^T t\,.\,dt$$

$$= \frac{A}{2}.$$

These are the exponential Fourier coefficients.

Exponential Fourier expansion is

$$\tilde{x}(t) = \frac{A}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{jA}{2\pi k} e^{jk\omega_0 t}$$

$$= \frac{A}{2} + j\frac{A}{2\pi}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k} e^{jk\omega_0 t} \qquad j\,\pi/2$$

(b) $$\tilde{x}(t) = \frac{A}{2} + \frac{A}{2\pi} \cdot$$

$$\sum_{k=1}^{\infty} \frac{j}{k} \left[e^{jk\omega_0 t} - e^{-jk\omega_0 t} \right] e^{j\pi/2}$$

$$= \frac{A}{2} - \frac{A}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{\left[e^{jk\omega_0 t} - e^{-jk\omega_0 t} \right]}{2j}$$

$$= \frac{A}{2} - \frac{A}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\omega_0 t).$$

4. (a) $x(t) = A \sin(\pi t), \quad 0 \le t \le 1$

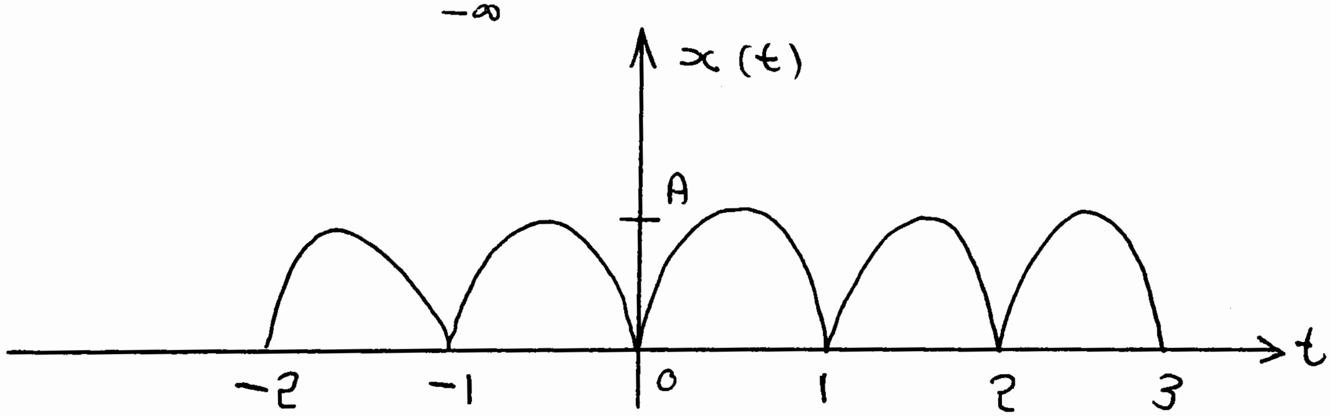
$$x(t) = x(t+T) \quad \text{where } T \triangleq 1.$$

Here fundamental angular frequency is

$$\omega_0 = \frac{2\pi}{T} = 2\pi.$$

Exponential Fourier series

$$\hat{x}(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$



Here

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_0^1 A \sin(\pi t)\, e^{-jk2\pi t}\, dt$$

$$\therefore \quad a_k = A \int_0^1 \frac{e^{j\pi t} - e^{-j\pi t}}{2j} e^{-jk2\pi t} \, dt$$

$$= \frac{A}{2j} \int_0^1 e^{-j\pi(2k-1)t} - e^{-j\pi(2k+1)t} \, dt$$

$$= \frac{A}{2j} \left[\frac{e^{-j\pi(2k-1)t}}{-j\pi(2k-1)} + \frac{e^{-j\pi(2k+1)t}}{j\pi(2k+1)} \right]_{t=0}^{t=1}$$

$$= \frac{A}{2j} \left[\frac{e^{-j\pi(2k+1)} - 1}{j\pi(2k+1)} - \frac{e^{-j\pi(2k-1)} - 1}{j\pi(2k-1)} \right]$$

$$= \frac{A}{2j} \left[\frac{e^{-j\pi} - 1}{j\pi(2k+1)} - \frac{e^{j\pi} - 1}{j\pi(2k-1)} \right]$$

(since $e^{-j\pi 2k} = 1$)

$$\therefore \quad a_k = \frac{A}{2j}\left[\frac{-2}{j\pi(2k+1)} + \frac{2}{j\pi(2k-1)}\right]$$

(since $e^{-j\pi} = e^{j\pi} = -1$)

$$\therefore \quad a_k = \frac{A}{\pi}\left(\frac{1}{(2k+1)} - \frac{1}{(2k-1)}\right)$$

$$= \frac{A}{\pi} \cdot \frac{(2k-1) - (2k+1)}{4k^2 - 1}$$

$$= \frac{-2A}{\pi} \cdot \frac{1}{4k^2-1}.$$

$\therefore$ exponential Fourier series is

$$x(t) = -\frac{2A}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1} \cdot e^{-jk2\pi t}$$

(b) clearly

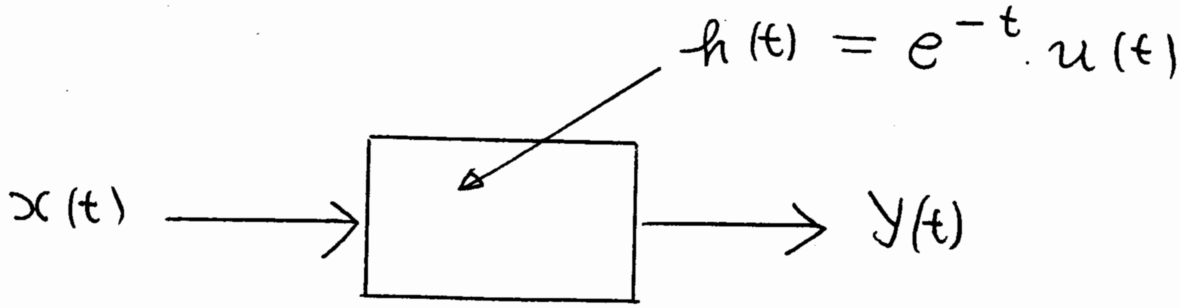
$$x(t) = \frac{2A}{\pi} - \frac{2A}{\pi} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{4k^2-1} \cdot e^{-jk2\pi t}$$

$$= \frac{2A}{\pi} - \frac{2A}{\pi} \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} \left[e^{-jk2\pi t} + e^{+jk2\pi t} \right]$$

$$= \frac{2A}{\pi} - \frac{4A}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \cdot \cos(k2\pi t).$$

(c)



$$(D+1)\, y(t) = x(t)$$

$\therefore \quad P(D) = 1 \qquad Q(D) = D + 1$

hence transfer function is

$$H(s) = \frac{1}{s+1}.$$

i.e. exponential Fourier series of the system response $y(t)$ is

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

and the exponential Fourier coefficients are

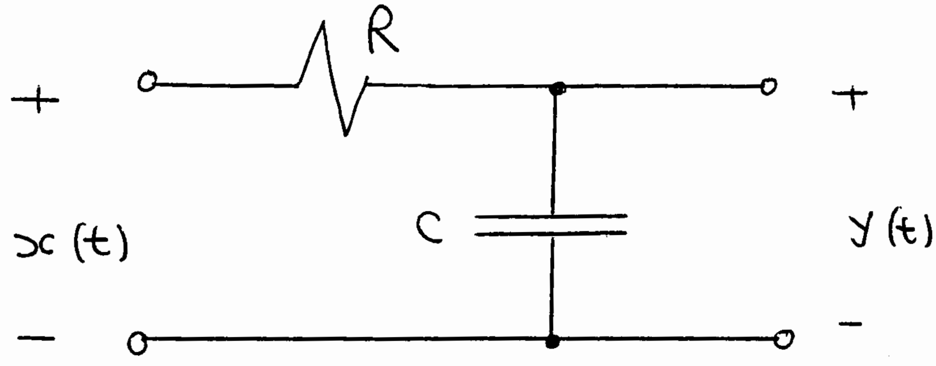
$$b_k = a_k H(jk\omega_0)$$

$$= a_k \cdot H(jk \cdot 2\pi)$$

$$= -\frac{2A}{\pi} \cdot \frac{1}{4k^2-1} \cdot \frac{1}{1+jk2\pi}.$$

5. 5.1

15.



$$x(t) = 100\cos(t) + 10\cos(3t) + \cos(5t).$$

Clearly the transfer function is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1 + sRC}$$

Exponential Fourier series for $x(t)$:

$$x(t) = 100\frac{[e^{jt} + e^{-jt}]}{2}$$

$$+ 10\frac{[e^{3jt} + e^{-3jt}]}{2}$$

$$+ \frac{[e^{5jt} + e^{-5jt}]}{2}$$

giving the exponential Fourier series

$$x(t) = \frac{1}{2}e^{-5jt} + 5e^{-3jt} + 50e^{-jt}$$

$$+ 50e^{jt} + 5e^{3jt} + \frac{1}{2}e^{5jt}$$

$$= \sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} a_k e^{jkt}$$

for

$$a_1 = a_{-1} = 50$$

$$a_3 = a_{-3} = 5$$

$$a_5 = a_{-5} = \frac{1}{2}$$

The response of [illegible] [illegible] [illegible]

i.e.

$$x(t) = \sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} a_k e^{jk\omega_0 t}$$

for $\omega_0 \triangleq 1.$

The corresponding output signal is

$$y(t) = \sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \underbrace{a_k H(jk\omega_0)}_{b_k} e^{jk\omega_0 t}$$

Put

$$b_k \triangleq a_k H(jk\omega_0)$$

$$\therefore \quad b_k = a_k \cdot \frac{1}{1 + jk\omega_0 RC}$$

$$= a_k \cdot \frac{1}{1 + jkRC}$$

$$= a_k \frac{1}{\sqrt{1 + k^2R^2C^2}} \angle \theta(k)$$

for

$$\theta_k \triangleq -\arctan(kRC).$$

Observe:

$$\theta_{-k} = -\theta_k$$

Thus

$$y(t) = b_1 e^{jt} + b_{-1} e^{-jt} + b_3 e^{3jt} + b_{-3} e^{-3jt} + b_5 e^{5jt} + b_{-5} e^{-5jt}$$

Now

$$b_1 e^{jt} + b_{-1} e^{-jt}$$

$$= \frac{a_1}{\sqrt{1+R^2c^2}} e^{jt} . e^{j\theta_1} + \frac{a_{-1}}{\sqrt{1+R^2c^2}} e^{-jt} e^{j\theta_{-1}}$$

$$= \frac{50}{\sqrt{1+R^2c^2}} \left[e^{j(t+\theta_1)} + e^{-j(t+\theta_1)} \right]$$

(since $\theta_{-1} = -\theta_1$)

$$= \frac{100}{\sqrt{1+R^2c^2}} \cos(t + \theta_1)$$

$$= \frac{100}{\sqrt{1+R^2c^2}} \cos(t - \tan^{-1}(RC)).$$

Similarly $b_3 e^{3jt} + b_{-3} e^{-3jt}$

$$= \frac{a_3}{\sqrt{1+9R^2C^2}} e^{3jt} \cdot e^{\theta_3} + \frac{a_{-3}}{\sqrt{1+9R^2C^2}} e^{-3jt} e^{\theta_{-3}}$$

$$= \frac{5}{\sqrt{1+9R^2C^2}} \left[e^{j(3t+\theta_3)} + e^{-j(3t-\theta_3)} \right]$$

(since $\theta_{-3} = -\theta_3$)

$$= \frac{10}{\sqrt{1+9R^2C^2}} \cos(3t + \theta_3)$$

$$= \frac{10}{\sqrt{1+9R^2C^2}} \cos\left[3t - \tan^{-1}(3RC)\right]$$

Likewise :

$$b_5 e^{5jt} + b_{-5} e^{-5jt}$$

$$= \frac{1}{\sqrt{1+25R^2C^2}} \cos\left[5t - \tan^{-1}(5RC)\right]$$

$$\therefore\ y(t) = \frac{100}{\sqrt{1+R^2C^2}} \cos\left[t - \tan^{-1}(RC)\right]$$

$$+ \frac{10}{\sqrt{1+9R^2C^2}} \cos\left[3t - \tan^{-1}(3RC)\right]$$

$$+ \frac{1}{\sqrt{1+25R^2C^2}} \cos\left[5t - \tan^{-1}(5RC)\right]$$

6. $x(t) = \cos^2(20\pi t) \cdot \sin(10\pi t)$ 6.1

We could use

$$\underline{P} = \frac{1}{T} \int_0^T |x(t)|^2 \, dt$$

for average power, but the integrations will certainly be tedious!

An easier approach is to use Parseval theorem.

First determine exponential Fourier series:

$$x(t) = \left[\frac{e^{j20\pi t} + e^{-j20\pi t}}{2} \right]^2 \cdot \left\{ \frac{e^{j10\pi t} - e^{-j10\pi t}}{2j} \right\}$$

$$= \frac{1}{8j} \left[e^{j40\pi t} + 2 + e^{-j40\pi t} \right] \left[e^{j10\pi t} - e^{-j10\pi t} \right]$$

6.2

25.

$$= \frac{1}{8j}\left[e^{j50\pi t} - e^{j30\pi t} + 2e^{j10\pi t} - 2e^{-j10\pi t} + e^{-j30\pi t} - e^{-j50\pi t}\right]$$

$$= \left(-\frac{1}{8j}\right)e^{-j50\pi t} + \left(\frac{1}{8j}\right)e^{-j30\pi t} + \left(-\frac{1}{4j}\right)e^{-j10\pi t} + \left(\frac{1}{4j}\right)e^{j10\pi t} - \left(\frac{1}{8j}\right)e^{j30\pi t} + \left(\frac{1}{8j}\right)e^{j50\pi t}$$

From Parseval theorem:

$$\underline{P} = \frac{1}{64}\left[1 + 1 + 4 + 4 + 1 + 1\right]$$

$$= \frac{3}{16}$$