ECE-342 Problem Set 6:Non-Periodic Inputs and Fourier Transforms

1. The signal $x(t)$ is band-limited by the angular frequency ω_B, namely its Fourier transform $X(\omega)$ is such that

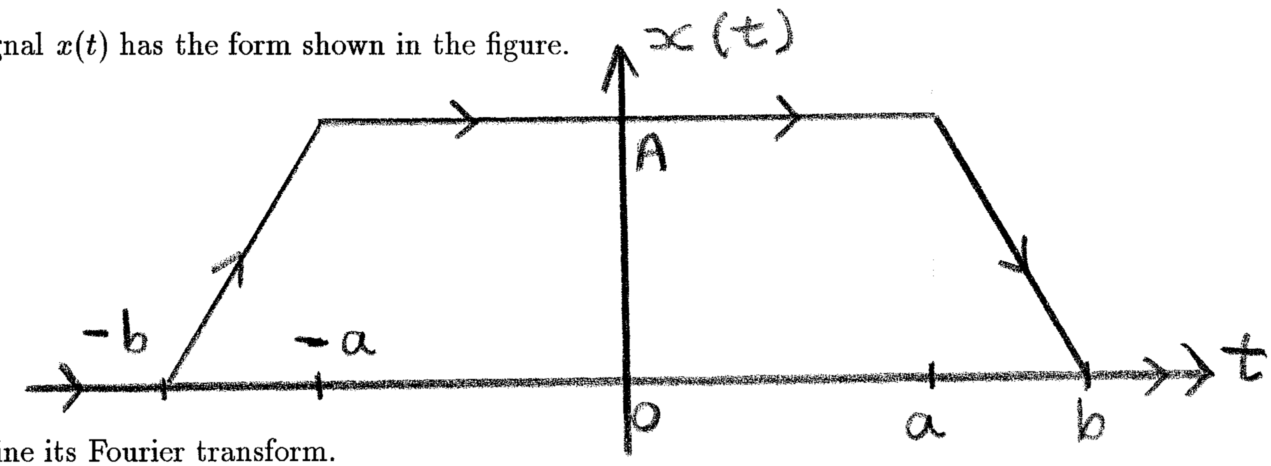
$$X(\omega) = 0, \qquad \text{for all } \omega \geq \omega_B,$$

and is the input to the ideal low-pass filter with transfer function

$$H(j\omega) = \begin{cases} Ce^{-j\omega_0 t}, & \text{for all } |\omega| \leq \omega_B, \\ 0, & \text{for all } |\omega| > \omega_B. \end{cases}$$

Determine the signal at the output of the filter.

2. A signal $x(t)$ has the form shown in the figure.



Determine its Fourier transform.

3. (a) Determine the Fourier transform of the signal

$$\operatorname{sgn}(t) \triangleq \begin{cases} 1, & \text{for all } t > 0, \\ 0, & \text{at } t = 0, \\ -1, & \text{for all } t < 0. \end{cases}$$

Hint: use the Fourier transform $\mathcal{F}\{u(t)\}(\omega)$.

(b) Use the result of (a) to determine the Fourier transform

$$\mathcal{F}\left\{\frac{1}{t}\right\}(\omega).$$

(c) The **Hilbert transform** of a signal $x(t)$ is defined to be the signal

$$\hat{x}(t) \triangleq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau.$$

Determine the Fourier transform of the Hilbert transform of $x(t)$, namely

$$\mathcal{F}\{\hat{x}(t)\}(\omega),$$

in terms of the Fourier transform $X(\omega)$ of $x(t)$.

4. Determine the Fourier transforms

(a) $\mathcal{F}\{e^{j\omega_0 t}u(t)\}(\omega)$.

(b) $\mathcal{F}\{\cos(\omega_0 t)u(t)\}(\omega)$.

(c) $\mathcal{F}\{\sin(\omega_0 t)u(t)\}(\omega)$.

5. Suppose that $h(t)$ is the impulse response of the causal LTI system, and let $H(j\omega)$ be its Fourier transform.

(a) Show that

$$h(t) = h_e(t) + \operatorname{sgn}(t)\, h_e(t),$$

where $h_e(t)$ is the even part of $h(t)$.

(b) Let

$$H_e(\omega) \triangleq \mathcal{F}\{h_e(t)\}(\omega),$$

and let

$$\hat{H}_e(\omega) \triangleq \int_{-\infty}^{\infty} \frac{H_e(\lambda)}{\omega - \lambda}\, d\lambda$$

be the Hilbert transform of $H_e(\omega)$. Show that

$$H(j\omega) = H_e(\omega) - \frac{j}{\pi}\hat{H}_e(\omega).$$

This result is essential in the study of **modulation systems** in communications.

1.

$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

Signal $x(t)$ is <u>band-limited</u> by angular frequency ω_B, namely

$$(1) \qquad X(\omega) = 0 \qquad \text{for all } |\omega| \geq \omega_B$$

$H(j\omega)$ is an ideal low-pass filter namely

$$(2) \qquad H(j\omega) = \begin{cases} c\, e^{-j\omega t_0}, & |\omega| \leq \omega_B \\ 0, & |\omega| > \omega_B. \end{cases}$$

Fourier transform of the output signal is

$$(3) \qquad Y(\omega) = H(j\omega)\, X(\omega)$$

thus output signal is

$$(4) \qquad y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega)\, e^{j\omega t}\, d\omega$$

From (2) and (3) we have

$$Y(\omega) = 0 \quad , \quad |\omega| > \omega_B$$

thus (4) becomes

$$y(t) = \frac{1}{2\pi} \int_{-\omega_B}^{\omega_B} Y(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_B}^{\omega_B} H(j\omega) X(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{c}{2\pi} \int_{-\omega_B}^{\omega_B} X(\omega) e^{j\omega t} \cdot e^{-j\omega t_0} d\omega$$

In view of (1) we can rewrite in the form

$$y(t) = \frac{c}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} e^{-j\omega t_0} d\omega$$

or

$$(5) \quad y(t) = \frac{c}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega(t-t_0)} d\omega$$

From Fourier's theorem we know

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

thus

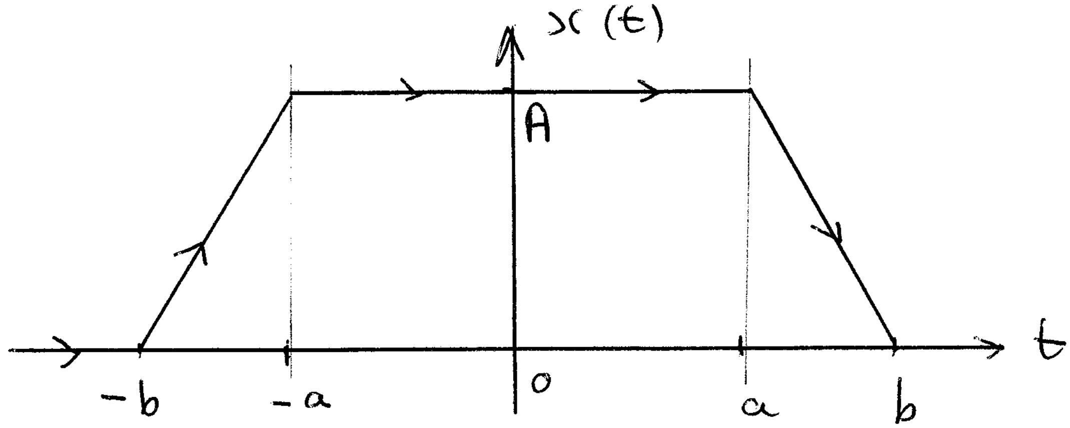
(6) $$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-t_0)}\, d\omega$$

use (6) in (5):

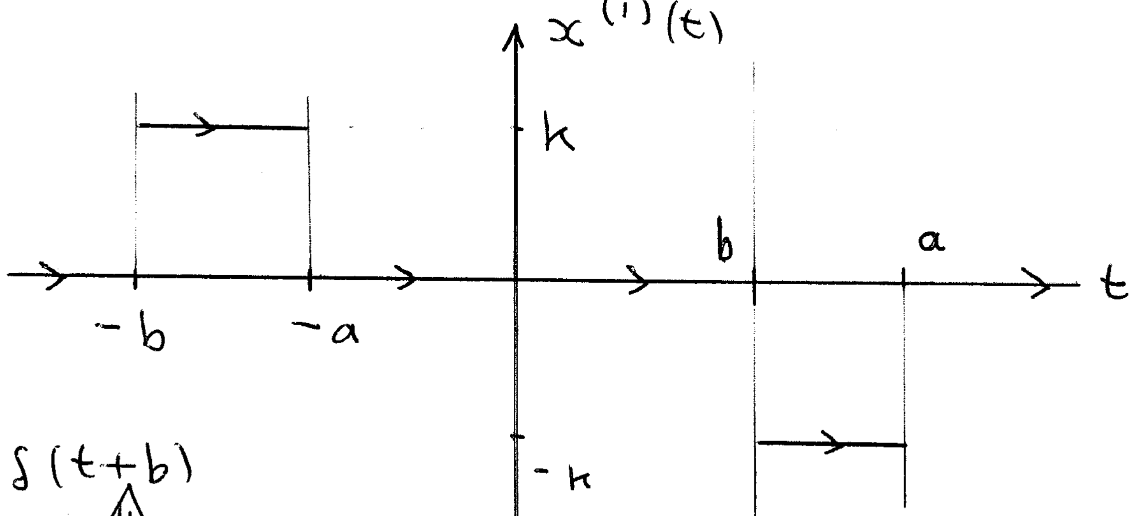
$$y(t) = c\, x(t-t_0).$$

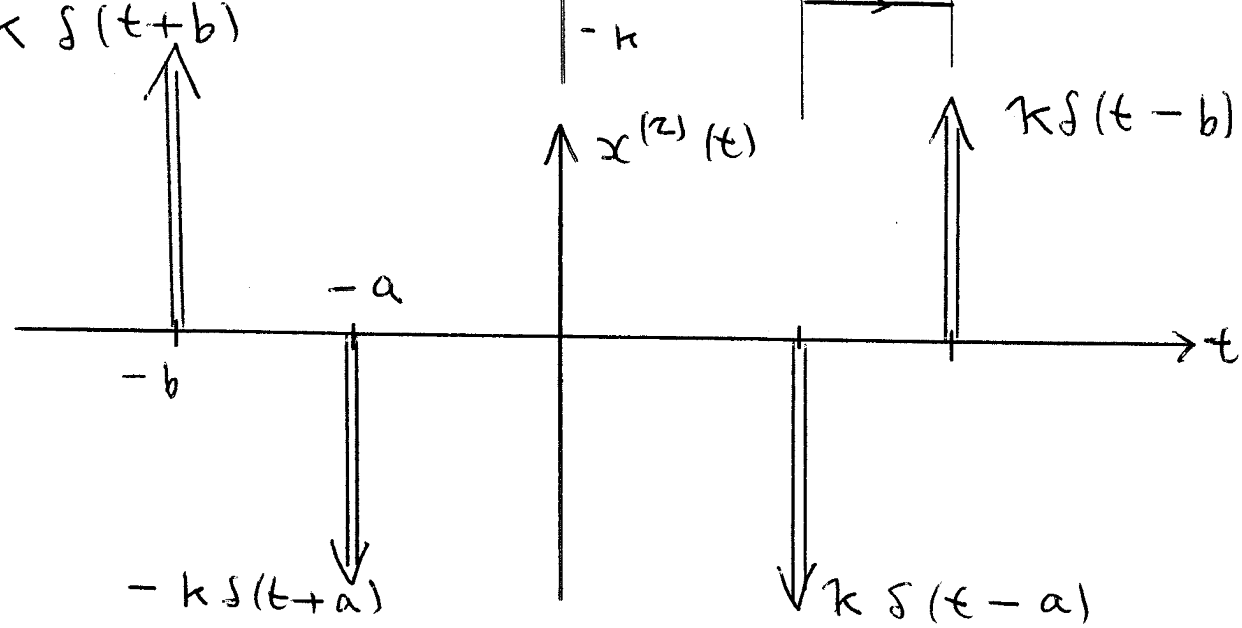
Conclusion: an ideal low-pass filter is a distortionless system for signals which are band-limited by ω_B, the band-width of the filter.

2.



One can use the definition of $X(\omega)$ and integrate directly. However the integrals are certain to be tedious. A better approach is to work out the signals $x^{(1)}(t)$ and $x^{(2)}(t)$:





Thus

$$x^{(2)}(t) = K\left[\delta(t+b) - \delta(t+a) - \delta(t-a) + \delta(t-b)\right]$$

where

$$K \triangleq (b-a)^{-1} A \qquad \text{— (1)}$$

The nice thing about $x^{(2)}(t)$ is that it is a sum of impulse functions, whose Fourier transforms are known, namely

$$\mathcal{F}\{\delta(t-t_0)\}(\omega) = e^{-j\omega t_0}$$

Use this in (1) gives:

$$\mathcal{F}\{x^{(2)}(t)\}(\omega)$$

$$= K\left[e^{j\omega b} - e^{j\omega a} - e^{-j\omega a} + e^{-j\omega b}\right]$$

$$= 2K\left[\cos(b\omega) - \cos(a\omega)\right] \qquad \text{— (2)}$$

Now

$$x^{(1)}(t) = \int_{-\infty}^{t} x^{(2)}(s)\,ds \quad \text{—— (3)}$$

By Property $(\overline{\text{VI}})$ of F - t forms

$$\mathcal{F}\{x^{(1)}(t)\}(\omega) \overset{(3)}{=} \mathcal{F}\left\{\int_{-\infty}^{t} x^{(2)}(s)\,ds\right\}(\omega)$$

$$= \frac{1}{j\omega}\mathcal{F}\{x^{(2)}(t)\}(\omega)$$

$$+ \pi\,\mathcal{F}\{x^{(2)}(t)\}(0)\,\delta(\omega)$$

(4)

From (2) with $\omega = 0$ have

$$\mathcal{F}\{x^{(2)}(t)\}(0) = 2k\,[\cos(0) - \cos(0)]$$

$$= 0 \quad \text{—— (5)}$$

Put (5) and (2) in (4)

$$\mathcal{F}\{x^{(1)}(t)\}(\omega) = \frac{2k}{j\omega}[\cos(b\omega) - \cos(a\omega)] + 0$$

(6)

Again

$$x(t) = \int_{-\infty}^{t} x^{(1)}(s)\,ds \quad \text{———} \quad (7)$$

Then

$$\mathcal{F}\{x(t)\}(\omega) \overset{(7)}{=} \mathcal{F}\left\{\int_{-\infty}^{t} x^{(1)}(s)\,ds\right\}(\omega)$$

$$\underset{\substack{\uparrow \\ \text{Property}(\overline{\text{VI}})}}{=} \frac{1}{j\omega}\mathcal{F}\{x^{(1)}(t)\}(\omega) + \underbrace{\pi\,\mathcal{F}\{x^{(1)}(t)\}(0)\,\delta(\omega)}_{(8)}$$

Must determine $\mathcal{F}\{x^{(1)}(t)\}(0)$.

Note that (6) with $\omega = 0$ gives the indeterminate

$$\mathcal{F}\{x^{(1)}(t)\}(0) = \frac{0}{0}$$

But from

$$\mathcal{F}\{x^{(1)}(t)\}(\omega) = \int_{-\infty}^{\infty} x^{(1)}(t)\,e^{-j\omega t}\,dt$$

with $\omega = 0$ get

$$\mathcal{F}\{x^{(1)}(t)\}(0) = \int_{-\infty}^{\infty} x^{(1)}(t)\, dt \qquad (9)$$

(since $e^0 = 1$)

From graph of $x^{(1)}(t)$ on page 2.1 see that <u>areas cancel</u> so that

$$\int_{-\infty}^{\infty} x^{(1)}(t)\, dt = 0$$

so that (9) gives

$$\mathcal{F}\{x^{(1)}(t)\}(0) = 0 \qquad (10).$$

Put (10) and (6) in (8):

$$\mathcal{F}\{x(t)\}(\omega) = \frac{2K[\cos(b\omega) - \cos(a\omega)]}{(j\omega)^2}$$

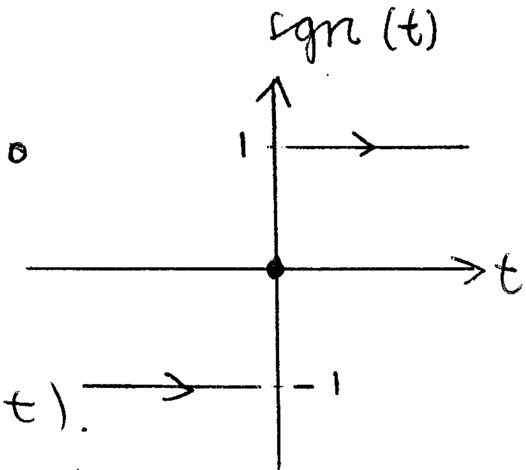
$$= \frac{2K[\cos(a\omega) - \cos(b\omega)]}{\omega^2}$$

(here $K \triangleq \dfrac{A}{b-a}$)

3. (a) $$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0 \\ -1, & t < 0. \end{cases}$$

Recalling that

$$u(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0 \end{cases}$$



we see that

$$\operatorname{sgn}(t) = u(t) - u(-t). \qquad (1)$$

Thus

$$\mathcal{F}\{\operatorname{sgn}(t)\}(\omega) = \mathcal{F}\{u(t)\}(\omega) - \mathcal{F}\{u(-t)\}(\omega) \qquad (2)$$

Now

$$\mathcal{F}\{u(t)\}(\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \qquad (3).$$

By symmetry of Fourier transforms and (3) we get

$$\mathcal{F}\{u(-t)\}(\omega) = \mathcal{F}\{u(t)\}(-\omega)$$

$$= \frac{1}{-j\omega} + \pi\delta(-\omega)$$

$$= \frac{-1}{j\omega} + \pi\delta(\omega) \quad \text{——} \quad (4)$$

Using (3) and (4) in (2):

$$\mathcal{F}\{\mathrm{sgn}(t)\}(\omega) = \frac{2}{j\omega}.$$

(b) By duality property of Fourier transforms:

$$\mathcal{F}\left\{\frac{2}{jt}\right\}(\omega) = 2\pi\,\mathrm{sgn}(-\omega)$$

$$= -2\pi\,\mathrm{sgn}(\omega)$$

(clearly sgn (.) is an <u>odd</u> function)

thus

$$\mathcal{F}\left\{\frac{1}{t}\right\}(\omega) = -\pi j\,\mathrm{sgn}(\omega).$$

(c) Hilbert transform of $x(t)$ is the signal

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \cdot d\tau$$

or

$$\hat{x}(t) = \frac{1}{\pi} \left(\frac{1}{t}\right) * x(t)$$

$$\therefore \mathcal{F}\{\hat{x}(t)\}(\omega)$$

$$= \frac{1}{\pi} \mathcal{F}\left\{\frac{1}{t}\right\}(\omega) . \mathcal{F}\{x(t)\}(\omega)$$

$$= -j \operatorname{sgn}(\omega) . X(\omega).$$

4. (a) We must find

$$\mathcal{F}\{e^{j\omega_0 t} \cdot u(t)\}(\omega).$$

Put

$$X_1(\omega) = \mathcal{F}\{e^{j\omega_0 t}\}(\omega)$$

$$X_2(\omega) = \mathcal{F}\{u(t)\}(\omega)$$

By multiplication property of Fourier transforms

$$\mathcal{F}\{e^{j\omega_0 t} \cdot u(t)\}(\omega)$$

$$= \frac{1}{2\pi}(X_1 * X_2)(\omega) \qquad \text{———} \qquad (1)$$

Now

$$X_1(\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$X_2(\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)$$

Thus $$(X_1 * X_2)(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\tau)\, X_1(\omega - \tau)\, d\tau$$

$$\therefore \quad (X_1 * X_2)(\omega)$$

$$= \frac{2\pi}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{j\tau} + \pi \delta(\tau) \right) \delta(\omega - \omega_0 - \tau) \, d\tau$$

$$= \frac{1}{j} \int_{-\infty}^{\infty} \frac{1}{\tau} \cdot \delta(\omega - \omega_0 - \tau) \, d\tau$$

$$+ \pi \int_{-\infty}^{\infty} \delta(\tau) \cdot \delta(\omega - \omega_0 - \tau) \, d\tau$$

$$= \frac{1}{j} \cdot \frac{1}{\omega - \omega_0} + \pi \cdot \delta(\omega - \omega_0).$$

(b) $\mathcal{F}\{ \cos(\omega_0 t) \cdot u(t) \}(\omega)$

$$= \left(\frac{1}{2}\right) \mathcal{F}\{ e^{j\omega_0 t} \cdot u(t) \}(\omega)$$

$$+ \left(\frac{1}{2}\right) \mathcal{F}\{ e^{-j\omega_0 t} \cdot u(t) \}(\omega)$$

$$= \frac{1}{2}\left(\frac{1}{j} \cdot \frac{1}{\omega - \omega_0} + \pi\, \delta(\omega - \omega_0) \right.$$

$$\left. + \frac{1}{j} \cdot \frac{1}{\omega + \omega_0} + \pi\, \delta(\omega + \omega_0) \right]$$

$$= \frac{\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right]$$

$$+ \frac{1}{j} \cdot \frac{\omega}{\omega^2 - \omega_0^2}.$$

5. (a) $h(t)$ is impulse response of a causal LTI system,

$\therefore$ $h(t) = 0$ when $t < 0$ ___ (1)

Introduce even and odd parts of $h(t)$:

$$h(t) = h_e(t) + h_o(t) \quad \text{\_\_\_} \quad (2)$$

where

$$h_e(t) \triangleq \frac{1}{2}[h(t) + h(-t)] \quad \text{\_\_\_} \quad (3)$$

$$h_o(t) \triangleq \frac{1}{2}[h(t) - h(-t)] \quad \text{\_\_\_} \quad (4)$$

when $t<0$: see from (1) and (2) that

$$h_e(t) + h_o(t) = 0$$

$\therefore$ $h_o(t) = -h_e(t)$ ___ (5)

when $t>0$: here it follows from (1)

that $h(-t) = 0$ thus from (3) and (4) we get

$$h_e(t) = \frac{1}{2} h(t)$$

$$h_o(t) = \frac{1}{2} h(t)$$

$$\therefore \quad h_o(t) = h_e(t) \quad \text{—} \quad (6)$$

combining (5) and (6) proves that

$$h_o(t) = \begin{cases} h_e(t), & t > 0, \\ -h_e(t), & t < 0 \end{cases}$$

or

$$h_o(t) = \operatorname{sgn}(t) \cdot h_e(t) \quad \text{—} \quad (7)$$

combine (7) and (2) gives

$$h(t) = h_e(t) + \operatorname{sgn}(t)\, h_e(t) \quad \text{—} \quad (8)$$

(b) Write

$$H_e(\omega) \triangleq \mathcal{F}\{h_e(t)\}(\omega) \quad \text{——} \quad (9)$$

Then from (8) and (9)

$$H(j\omega) = \mathcal{F}\{h(t)\}(\omega)$$

$$= H_e(\omega) + \mathcal{F}\{\mathrm{sgn}(t)\, h_e(t)\}(\omega) \quad \text{——} \quad (10)$$

Now put

$$I(\omega) \triangleq \mathcal{F}\{\mathrm{sgn}(t)\}(\omega).$$

From Problem 3 (a)

$$I(\omega) = \frac{2}{j\omega} \quad \text{——} \quad (11)$$

By multiplication theorem we get

$$\mathcal{F}\{\mathrm{sgn}(t)\, h_e(t)\}(\omega) = \frac{(I * H_e)(\omega)}{2\pi}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H_e(\lambda)\, I(\omega - \lambda)\, d\lambda$$

$$= \frac{1}{2\pi} \frac{2}{j} \int_{-\infty}^{\infty} \frac{H_e(\lambda)}{\omega - \lambda} \cdot d\lambda$$

$$= -\frac{j}{\pi} \hat{H}_e(\omega) \quad \text{——} \quad (12)$$

where
$$\hat{H}_e(\omega) \triangleq \int_{-\infty}^{\infty} \frac{H_e(\lambda)}{\omega - \lambda} \cdot d\lambda$$

(In the terminology of Problem 3(c) we call $\hat{H}_e(\omega)$ the Hilbert transform of $H_e(\omega)$)

Thus:

$$H(j\omega) = H_e(\omega) - \frac{j}{\pi} \hat{H}_e(\omega).$$