UNIVERSITY OF WATERLOO

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

E& CE 342 Signals and Systems

Midterm Examination

Thursday, June 17, 1999, 4.30 pm - 6.30 pm

Instructor: A.J. Heunis
Time of exam: 4.30 p.m.
Duration of exam: 90 minutes
Aids permitted: Hand calculators only
Answer all questions
Total marks = 60
Each question carries 15 marks
Mark allocation within questions is shown

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1. (a)[10] A linear time-invariant system has response to a unit step input $x_0(t) \triangleq u(t)$ given by

$$y_0(t) = \begin{cases} 0, & \text{when } t < 0, \\ t^2, & \text{when } t \geq 0. \end{cases}$$

Now consider the input signal

$$x(t) = \begin{cases} 0, & \text{when } t < -1, \\ -2, & \text{when } -1 \leq t < 0, \\ 1, & \text{when } 0 \leq t < 1, \\ 2, & \text{when } 1 \leq t < 2, \\ 0, & \text{when } 2 \leq t. \end{cases}$$

(ii) Write $x(t)$ as a linear combination of shifted unit step functions.

(iii) Write the the response $y(t)$ of the system to the input signal $x(t)$ as a linear combination of shifted functions $y_0(t)$. Then write $y(t)$ as a function of t (leave your answer in unsimplified form).

(b) [5] A signal $x(t)$ has the form

$$x(t) = \begin{cases} 0, & \text{for all } t < 0, \\ \sin(t), & \text{for all } t \geq 0. \end{cases}$$

Determine the even and odd parts of $x(t)$.

2. (a) [9] Deduce whether the following systems are linear or not. In each case carefully explain the reasoning to justify your conclusion.

(i) $Dy(t) + y(t) = \cos(x(t))$, $\quad y(0) = 0$.
(ii) $y(t) = t[x(t)]^2$.
(iii) In the system of Fig. 2 the input signal is the applied current $x(t)$, and the output signal is the corresponding voltage $y(t)$ across the terminals A and B.

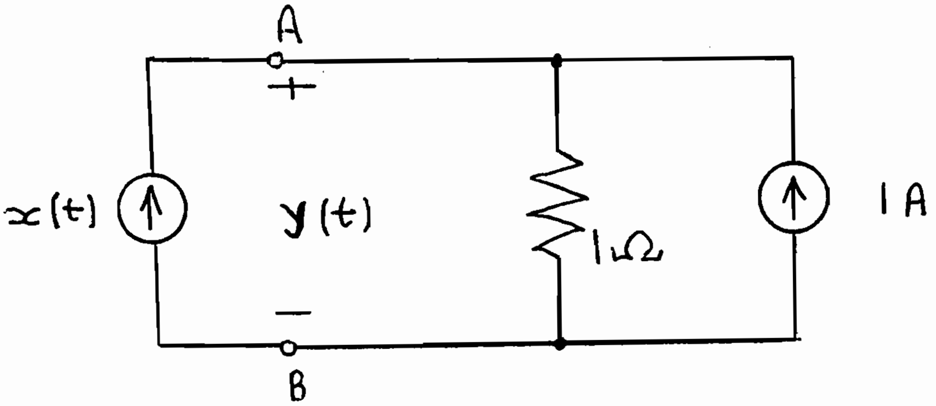


Fig. 2.

(b) [6] Deduce whether the following systems are time-invariant or not. In each case carefully explain the reasoning to justify your conclusion.

(i) $y(t) = t[x(t)]^2$.
(ii) $y(t) = [x(2t)]^2$.

3. A linear time-invariant system which is initially at rest has impulse response given by

$$h(t) = \begin{cases} 1, & \text{for all } -1 \leq t < 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) [12] Determine the response of the system to the input signal

$$x(t) = \begin{cases} 0, & \text{for all } t < 0, \\ e^{2t}, & \text{for all } t \geq 0. \end{cases}$$

(b) [3] Deduce whether the system is causal or not.

4. In the circuit shown in Fig. 4 the operational amplifier is ideal.

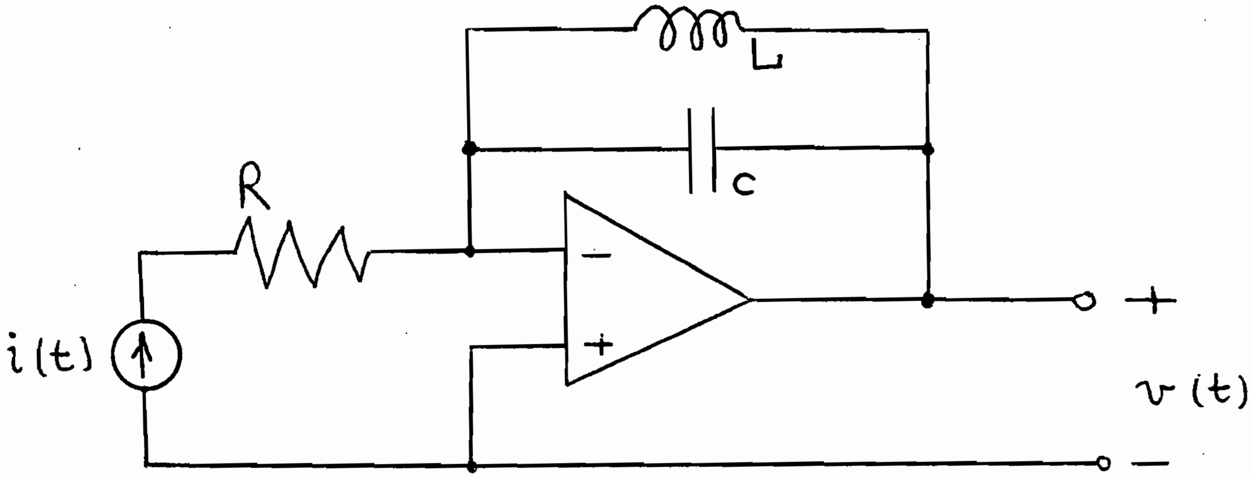


Fig. 4.

(a) [12] The input signal is the applied current $i(t)$ and the output signal is the corresponding voltage $v(t)$. Determine a relation of the form

$$Q(D)v(t) = P(D)i(t)$$

and clearly identify the polynomials $Q(D)$ and $P(D)$.

(b) [3] Determine whether or not the system is BIBO stable. Justify your conclusion.

USEFUL FACTS:

Trig. formulae: For α in radians we have

$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}, \qquad \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}.$$

Energy in signals:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt, \qquad E = \sum_{k=-\infty}^{\infty} |x[k]|^2.$$

Average power in signals:

$$P = \lim_{a\to\infty} \frac{1}{2a}\int_{-a}^{a} |x(t)|^2 dt, \qquad P = \lim_{n\to\infty} \frac{1}{2n+1}\sum_{k=-n}^{n} |x[k]|^2.$$

Even and odd parts of a signal:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)], \qquad x_o(t) = \frac{1}{2}[x(t) - x(-t)],$$

Sifting formula: For a continuous signal $x(t)$ we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

Convolution of two signals:

$$(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau))dt.$$

Output of a linear system: A linear time-invariant system with impulse response $h(t)$ and initially at rest, has response to an input $x(t)$ given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)dt.$$

Zero-input response: Consider the linear system

$$\begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

with

$$Q(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\ldots(\lambda-\lambda_r)^{m_r},$$

Case 1: The roots λ_i are distinct. The zero-input response is given by

$$y_{zi}(t) \triangleq \sum_{i=1}^{n} c_i e^{\lambda_i t},$$

where constants c_i are determined from the initial conditions.

Case 2: Some roots λ_i are repeated. The zero-input response is given by

$$y_{zi}(t) \triangleq \sum_{i=1}^{r}\sum_{j=1}^{m_i} c_{i,j} t^{j-1} e^{\lambda_i t},$$

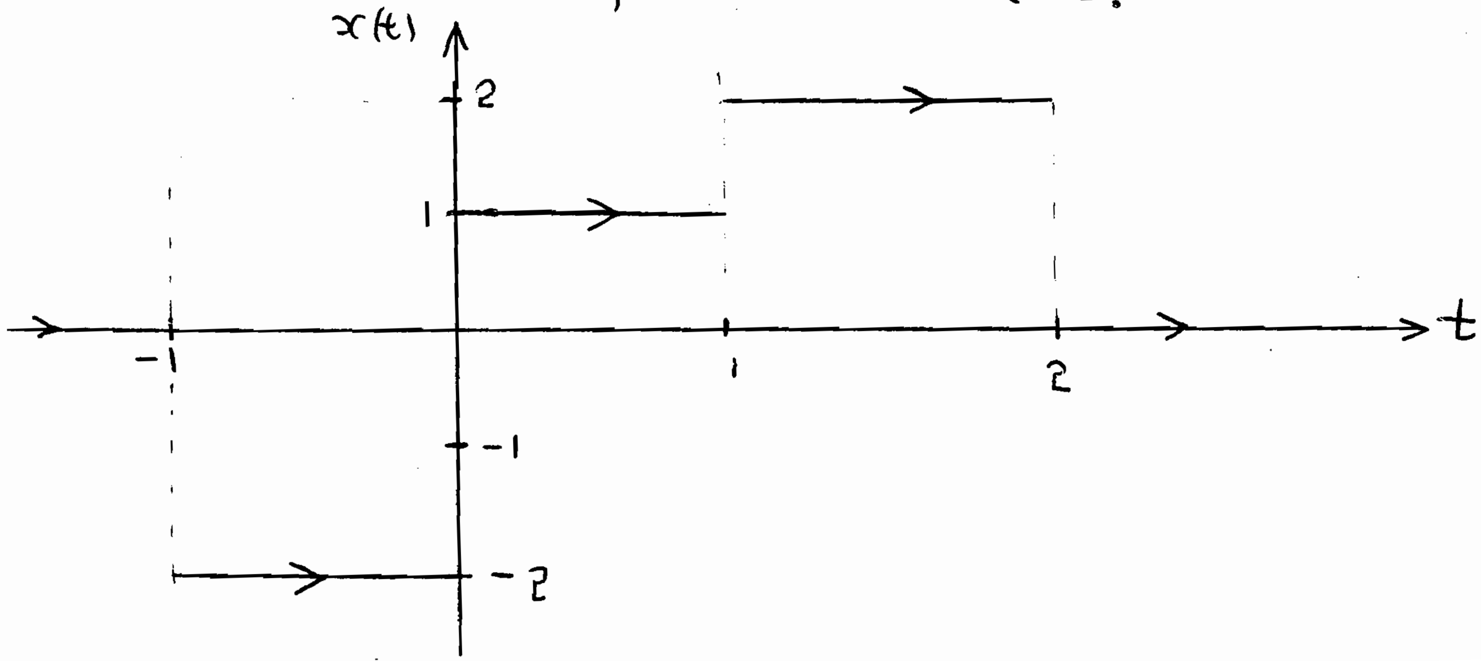
where constants $c_{i,j}$ are determined from the initial conditions.

1(a) Unit-step input $x_0(t) \triangleq u(t)$ causes response

$$y_0(t) = \begin{cases} 0, & t < 0 \\ t^2, & t \geq 0. \end{cases}$$

Now consider input signal

$$\begin{aligned} x(t) &= 0, & t < -1, \\ &= -2, & -1 \leq t < 0, \\ &= 1, & 0 \leq t < 1, \\ &= 2, & 1 \leq t < 2 \\ &= 0, & 2 \leq t. \end{aligned}$$



(i) clearly we can "make" $x(t)$ from unit step functions $x_0(t)$ as follows:

$$x(t) = -2x_0(t+1) + 3x_0(t) + x_0(t-1) - 2x_0(t-2).$$

(ii) By linearity and time-invariance the response $y(t)$ to the signal $x(t)$ is

$$y(t) = -2y_0(t+1) + 3y_0(t) + y_0(t-1) - 2y_0(t-2)$$

Now

$$y_0(t+1) = \begin{cases} 0, & t < -1 \\ (t+1)^2 & t \geq -1 \end{cases}$$

thus

$$y_0(t+1) = (t+1)^2 \cdot u(t+1)$$

also

$$y_0(t) = t^2 \cdot u(t)$$

$$y_0(t-1) = (t-1)^2 u(t-1)$$

$$y_0(t-2) = (t-2)^2 u(t-2)$$

thus

$$y(t) = -2(t+1)^2 u(t+1) + 3t^2 u(t)$$

$$+ (t-1)^2 u(t-1) - 2(t-2)^2 \cdot u(t-2).$$

(b) $x(t) = \sin(t).\ u(t).$

even part :

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$= \frac{\sin(t).\ u(t) + \sin(-t).\ u(-t)}{2}$$

$$= \frac{\sin(t).u(t) - \sin(t).\ u(-t)}{2}$$

$$= \frac{\sin(t)\left[u(t) - u(-t)\right]}{2}$$

$$= \begin{cases} \frac{1}{2}\sin(t), & t \geq 0 \\ -\frac{1}{2}\sin(t), & t < 0. \end{cases}$$

odd part :

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$= \frac{1}{2}\left[\sin(t) u(t) - \sin(-t). u(-t)\right]$$

$$= \frac{\sin(t)\,u(t) + \sin(t)\cdot u(-t)}{2}$$

$$= \frac{1}{2}\sin(t)\,[u(t) + u(-t)]$$

$$= \frac{1}{2}\sin t.$$

2. (a)

(i) $Dy(t) + y(t) = \cos[x(t)]$ ____ (1)

Suppose system is linear. Then

$x(t)$ zero signal $\Rightarrow$ $y(t)$ zero signal.

But, if $x(t)$ in (1) is zero signal then

$$\cos(x(t)) = 1 \qquad \text{for all } t$$

thus

$$Dy(t) + y(t) = 1, \quad \text{all } t,$$

which contradicts fact that $y(t)$ is zero signal.

Thus system cannot be linear.

(ii) $y(t) = t\,[x(t)]^2$

Take input
$$x_1(t) = 1 \qquad \text{for all } t.$$

Thus
$$y_1(t) = t. \qquad \text{"} \quad \text{"} \quad \text{—— (1)}$$

Take input
$$x_2(t) = -1 \qquad \text{for all } t$$

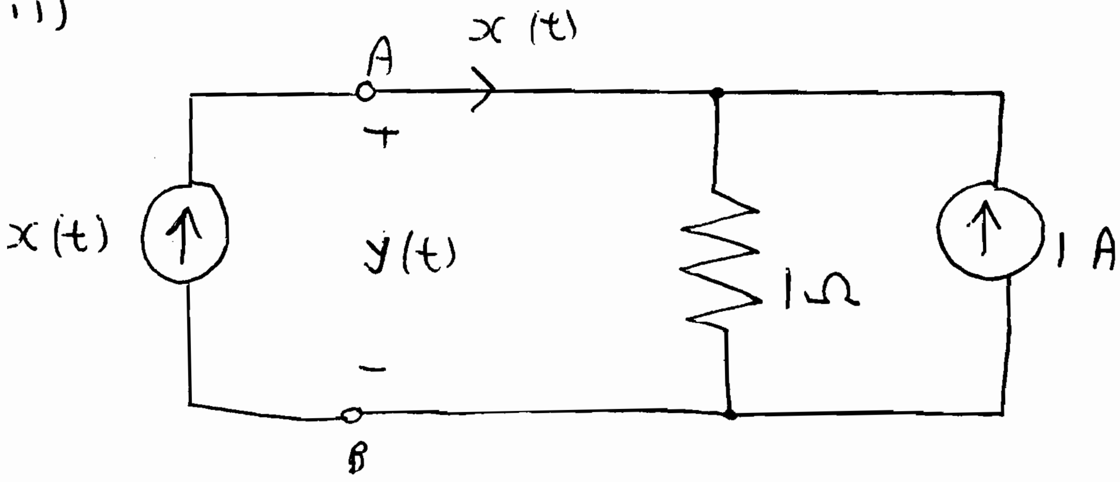
thus
$$y_2(t) = t \qquad \text{"} \quad \text{"} \quad \text{—— (2)}$$

If system is linear then
$$y_1(t) = -y_2(t)$$

which contradicts (1), (2).

$\therefore$ the system is not linear.

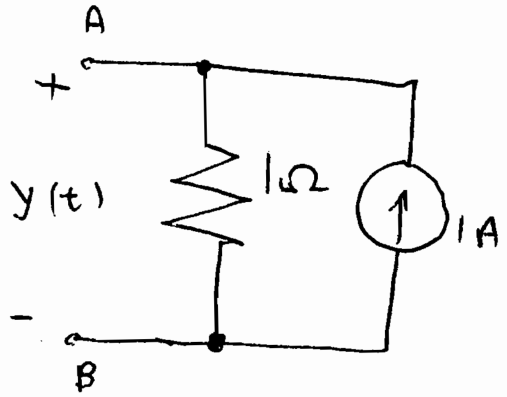
(iii)



Suppose $x(t)$ is the zero signal.

Then current source to left of A - B becomes an open circuit. Thus output signal is

$$y(t) = 1 \text{ volt.} \tag{1}$$



But, if system were linear, then output $y(t)$ which corresponds to the zero input signal, is also the zero signal.

Thus (1) shows that system cannot be linear.

(b) (i) $y(t) = t\,[x(t)]^2$

Let $y_1(t)$ be response to input signal $x_1(t)$. Fix instant $t_0 \neq 0$. Let $y_2(t)$ be response to input signal

$$x_2(t) \triangleq x_1(t - t_0)$$

namely

$$y_2(t) = t\,[x_2(t)]^2$$

$$= t\,[x_1(t - t_0)]^2 \quad \text{---} \quad (1)$$

also clearly

$$y_1(t) = t\,[x_1(t)]^2$$

so that

$$y_1(t - t_0) = (t - t_0)\,[x_1(t - t_0)]^2 \quad \text{---} \quad (2)$$

Comparing (1) and (2) we see that

$$y_1(t - t_0) \neq y_2(t)$$

so that the system <u>cannot</u> be time - invariant. 2.

(ii) $y(t) = [x(2t)]^2$

Let $y_1(t)$ be response to input signal $x_1(t)$.

Fix $t_0 \neq 0$.

Let $y_2(t)$ be response to input signal

$$x_2(t) \triangleq x_1(t - t_0) \quad \text{\_\_\_\_} \quad (1)$$

From (1) we have

$$x_2(2t) = x_1(2t - t_0)$$

thus

$$y_2(t) = [x_2(2t)]^2$$

$$= [x_1(2t - t_0)]^2 \quad \text{\_\_\_\_} \quad (2)$$

also clearly

$$y_1(t) = [x_1(2t)]^2$$

so that

$$y_1(t - t_0) = [x_1(2(t - t_0))]^2$$

$$= [x_1(2t - 2t_0)]^2 \quad \text{———} \quad (3)$$

Comparing (2) and (3) shows that generally

$$y_1(t - t_0) \neq y_2(t)$$

so that system cannot be time - invariant.

3 $$h(t) = \begin{cases} 1, & -1 \leq t \leq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$x(t) = \begin{cases} 0, & t \leq 0 \\ e^{2t}, & t > 0. \end{cases}$$

(a) The system is initially at rest, hence

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{—— (1)}$$

Now clearly

$$x(\tau) = e^{2\tau}\, I[0,\infty)(\tau) \quad \text{—— (2)}$$

Also

$$h(t-\tau) = 1 \quad \text{iff} \quad -1 \leq t-\tau \leq 0 \quad \text{—— (3)}$$

Now

$$-1 \leq t - \tau \leq 0$$

iff $\quad -1 \leq t - \tau \quad$ and $\quad t - \tau \leq 0$

iff $\quad \tau \leq t + 1 \quad$ and $\quad t \leq \tau$

iff $\quad t \leq \tau \leq t + 1.$

Thus

$$h(t - \tau) = I[t, t+1](\tau) \quad \_\_ \quad (4)$$

By (2), (4)

$x(\tau)\, h(t - \tau)$

$$= e^{2\tau} I[0, \infty)(\tau)\, I[t, t+1](\tau)$$

$$= e^{2\tau} I[\max\{0, t\}, \min\{t+1, \infty\}](\tau)$$

$$= e^{2\tau} I[\alpha(t), \beta(t)](\tau) \quad \_\_ \quad (5)$$

where

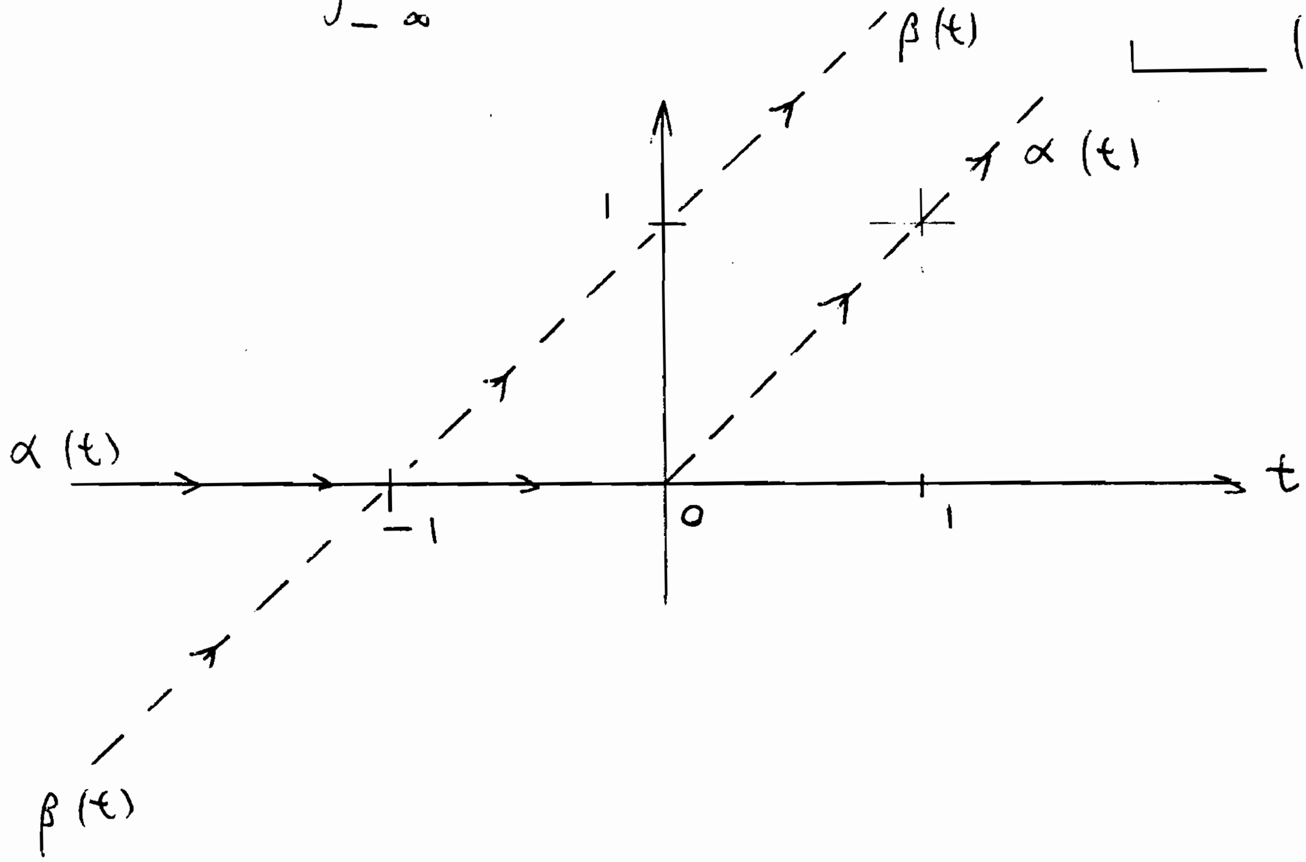
$$\alpha(t) \triangleq \max\{0, t\} \quad \text{———} \quad (6)$$

$$\beta(t) \triangleq \min\{t+1, \infty\}$$

$$= t+1 \quad \text{———} \quad (7)$$

From (1) and (5)

$$y(t) = \int_{-\infty}^{\infty} e^{2\tau}\, I[\alpha(t), \beta(t)](\tau)\, d\tau \quad \text{———} \quad (8)$$



$t < -1$ here $\alpha(t) > \beta(t)$

thus

$$I\,[\alpha(t),\ \beta(t)](\tau) \equiv 0 \quad \text{all } \tau$$

hence from (8)

$$y(t) = 0.$$

$-1 \leqslant t \leqslant 0$ here $\alpha(t) = 0$

$$\beta(t) = t+1.$$

From (8)

$$y(t) = \int_{\alpha(t)}^{\beta(t)} e^{2\tau}\, d\tau = \int_0^{t+1} e^{2\tau}\, d\tau$$

$$= \frac{e^{2\tau}}{2}\Bigg|_0^{t+1}$$

$$= \frac{1}{2}\left[e^{2(t+1)} - 1\right]$$

$\underline{0 \leq t}$ here $\alpha(t) = t$

$$\beta(t) = t+1$$

From (8)

$$y(t) = \int_t^{t+1} e^{2\tau}\, d\tau$$

$$= \frac{e^{2\tau}}{2} \Bigg|_{\tau=t}^{\tau=t+1}$$

$$= \frac{1}{2}\left[e^{2(t+1)} - e^{2t}\right]$$

$$= \frac{e^{2t}}{2}\left[e^{2} - 1\right].$$

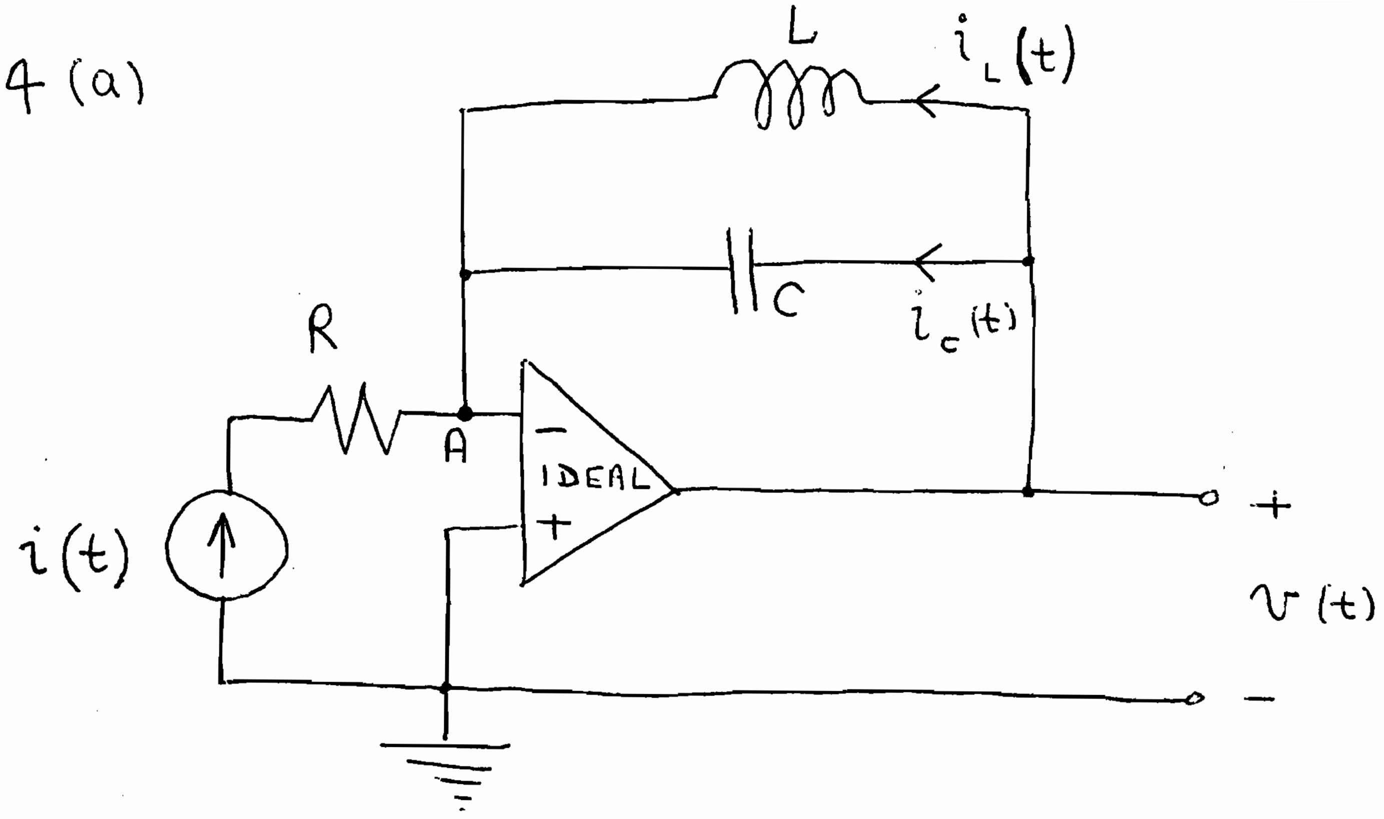
(b) $$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

From (4):

$$y(t) = \int_{t}^{t+1} x(\tau)\, d\tau$$

This shows the system is <u>non</u>-causal.

4 (a)



Introduce datum node shown.

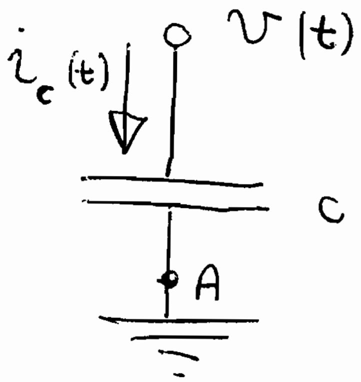
KCL at node A:

$$i(t) + i_L(t) + i_c(t) = 0 \quad \text{——} \quad (1)$$

Now

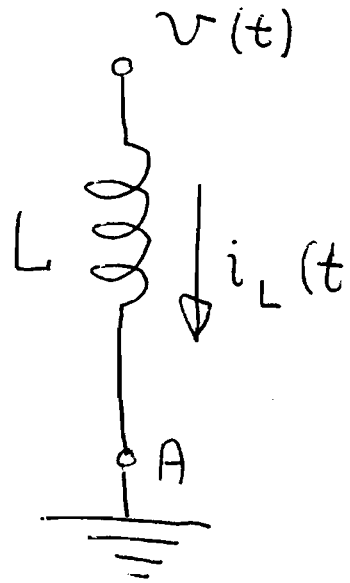
$$v_A(t) \equiv 0 \qquad \text{for all } t \quad \text{——} \quad (2)$$

(using properties of ideal op amps)



By capacitor equation

$$i_c(t) = C\dot{v}(t) \quad \_\_\_\_ \quad (3)$$



By inductor equation

$$v(t) = L \frac{d}{dt} i_L(t)$$

hence

$$i_L(t) = i_L(0) + \frac{1}{L}\int_0^t v(\tau)\, d\tau \quad \_\_ \quad (4)$$

Combine (1), (3), (4)

$$i(t) + \left\{ i_L(0) + \frac{1}{L}\int_0^t v(\tau)\, d\tau \right\} + C\dot{v}(t) = 0$$

Take derivatives wrt t

$$Di(t) + \frac{1}{L} v(t) + C D^2 v(t) = 0$$

or

$$D^2 v(t) + \frac{1}{CL} v(t) = -\frac{1}{C} Di(t)$$

Hence

$$Q(D) = D^2 + \frac{1}{LC}$$

$$P(D) = -\frac{1}{C} D$$

(b) Roots of $Q(\lambda)$ are

$$\lambda_1 = \frac{j}{\sqrt{LC}} \qquad \lambda_2 = \frac{-j}{\sqrt{LC}}$$

hence

$$re(\lambda_1) = re(\lambda_2) = 0.$$

Thus system fails to be both asymp. stable and BIBO stable.