UNIVERSITY OF WATERLOO
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
ECE 342 (Section 1) Signals and Systems
Midterm Examination
Thursday June 19, 2008, 5.30 pm - 6.45 pm

Instructor: A.J. Heunis
Time of exam: 5.30 p.m.
Duration of exam: 75 minutes
Aids permitted: Hand calculators only
Answer all three questions
Total marks = 60
Each question carries 20 marks
Mark allocation within questions is shown in square brackets

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1. A linear time-invariant system has the impulse response $h(t)$ given by

$$h(t) = \begin{cases} 1, & \text{when } -1 \leq t \leq +1 \,, \\ 0, & \text{for all other values of } t. \end{cases}$$

(i) [4] is the system causal ? (briefly discuss - 4 lines at most!).
(ii) [16] determine the response to the input signal $x(t)$ given by

$$x(t) = \begin{cases} 1, & \text{when } -1 \leq t \leq +1 \,, \\ 0, & \text{for all other values of } t. \end{cases}$$

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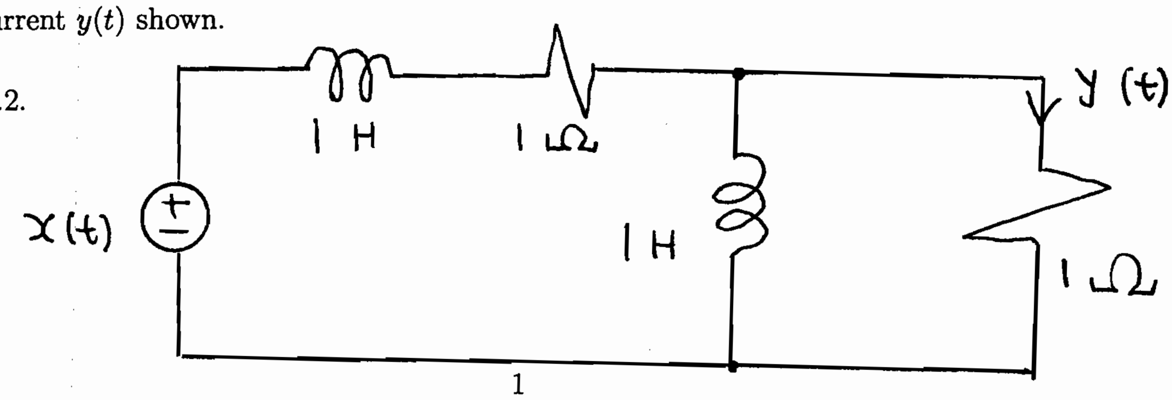
2. For the circuit in Fig. Qu.2 the input signal is the applied voltage $x(t)$ and the output signal is the current $y(t)$ shown.

Fig. Qu.2.



Qu. 2 continues ...

(i) [10] determine the linear differential equation that relates the input and output signals.
(ii) is the system
  (a) [4] BIBO stable ?
  (b) [3] linear for the initial conditions $y(0-) = 1$ and $y^{(1)}(0-) = 1$?
In each case give reasons for your answer.

(iii) [3] Suppose that $x(t)$ is the unit-step function. Explain **very briefly** the effect on the output $y(t)$ at $t = 0$. (do NOT try to calculate $y(t)$!!)

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3. In the following make sure that your reasoning is **clear and conclusive** in each case!

(i) [7] The input signal $x(t)$ and output signal $y(t)$ of a system are related by

$$y(t) = 2 + \int_{-\infty}^{\frac{t}{4}-3} x(\tau)\, d\tau, \qquad \text{for all } t.$$

Determine whether or not the system is causal.

(ii) [7] The input signal $x(t)$ and output signal $y(t)$ are related by

$$y(t) = x(at), \qquad \text{for all } t,$$

in which a is a constant, either positive or negative in value. Determine all possible values of the constant a for which the system is causal.

(iii) [6] The input signal $x(t)$ and output signal $y(t)$ are related by

$$y(t) = x(t^3), \qquad \text{for all } t.$$

Determine whether or not the system is time-invariant.

USEFUL FACTS:

Trig. formulae: For α in radians we have

$$\cos\alpha = \frac{e^{j\alpha}+e^{-j\alpha}}{2}, \qquad \sin\alpha = \frac{e^{j\alpha}-e^{-j\alpha}}{2j}.$$

Energy in signals:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt, \qquad E = \sum_{k=-\infty}^{\infty} |x[k]|^2.$$

Average power in signals:

$$P = \lim_{a\to\infty} \frac{1}{2a}\int_{-a}^{a} |x(t)|^2 dt, \qquad P = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n} |x[k]|^2.$$

Even and odd parts of a signal:

$$x_e(t) = \frac{1}{2}[x(t)+x(-t)], \qquad x_o(t) = \frac{1}{2}[x(t)-x(-t)],$$

Sifting formula: For a signal $x(t)$ with n continuous derivatives we have

$$x^{(n)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(n)}(t-\tau)d\tau, \qquad n = 0,1,2,\ldots$$

Convolution of two signals:

$$(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau))d\tau.$$

Output of a linear system: A linear time-invariant system has impulse response $h(t)$. Then the response to the input $x(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)dt.$$

Zero-input response: Consider the system

$$\begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

with

$$Q(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\ldots(\lambda-\lambda_r)^{m_r},$$

Case 1: The roots λ_i are distinct. The zero-input response is given by

$$y_{zi}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t},$$

where constants c_i are determined from the initial conditions.

Case 2: Some roots λ_i are repeated. The zero-input response is given by

$$y_{zi}(t) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} c_{i,j} t^{j-1} e^{\lambda_i t},$$

where constants $c_{i,j}$ are determined from the initial conditions.

1. $$h(t) = \begin{cases} 1, & -1 \leq t \leq +1, \\ 0, & \text{otherwise.} \end{cases} \quad \text{—— (1)}$$

(i) from section 2.2.1 of notes we know

system is causal <u>iff</u> $h(t) = 0$ all $t < 0$ —— (2)

(for an LTI system)

From (1) we <u>do NOT</u> have $h(t) = 0$ for all $t < 0$ (since $h(t) = 1$ for $-1 \leq t < 0$)

so from (2)

system is <u>NOT</u> causal.

(ii) $$x(t) = \begin{cases} 1, & -1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{—— (3)}$$

Response given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{—— (4)}$$

From 2

$$x(t) = I[-1, 1](t) \quad \text{--- } 5$$

From 1

$$h(t) = I[-1, 1](t) \quad \text{--- } 6$$

For fixed t have from 1

$$h(t-\tau) = \begin{cases} 1, & -1 \leq t-\tau \leq 1, \\ 0, & \text{otherwise} \end{cases} \quad \text{--- } 7$$

But

$$-1 \leq t - \tau \leq 1$$

<u>iff</u> $\quad -t-1 \leq -\tau \leq -t+1$

<u>iff</u> $\quad t+1 \geq \tau \geq t-1 \quad \text{--- } 8$

From 7 8 for fixed t have

$$h(t-\tau) = \begin{cases} 1, & t-1 \leq \tau \leq t+1, \\ 0, & \text{otherwise} \end{cases}$$

so that

$$h(t-\tau) = I[t-1, t+1](\tau) \quad \text{--- (9)}$$

From (9) (5) (4)

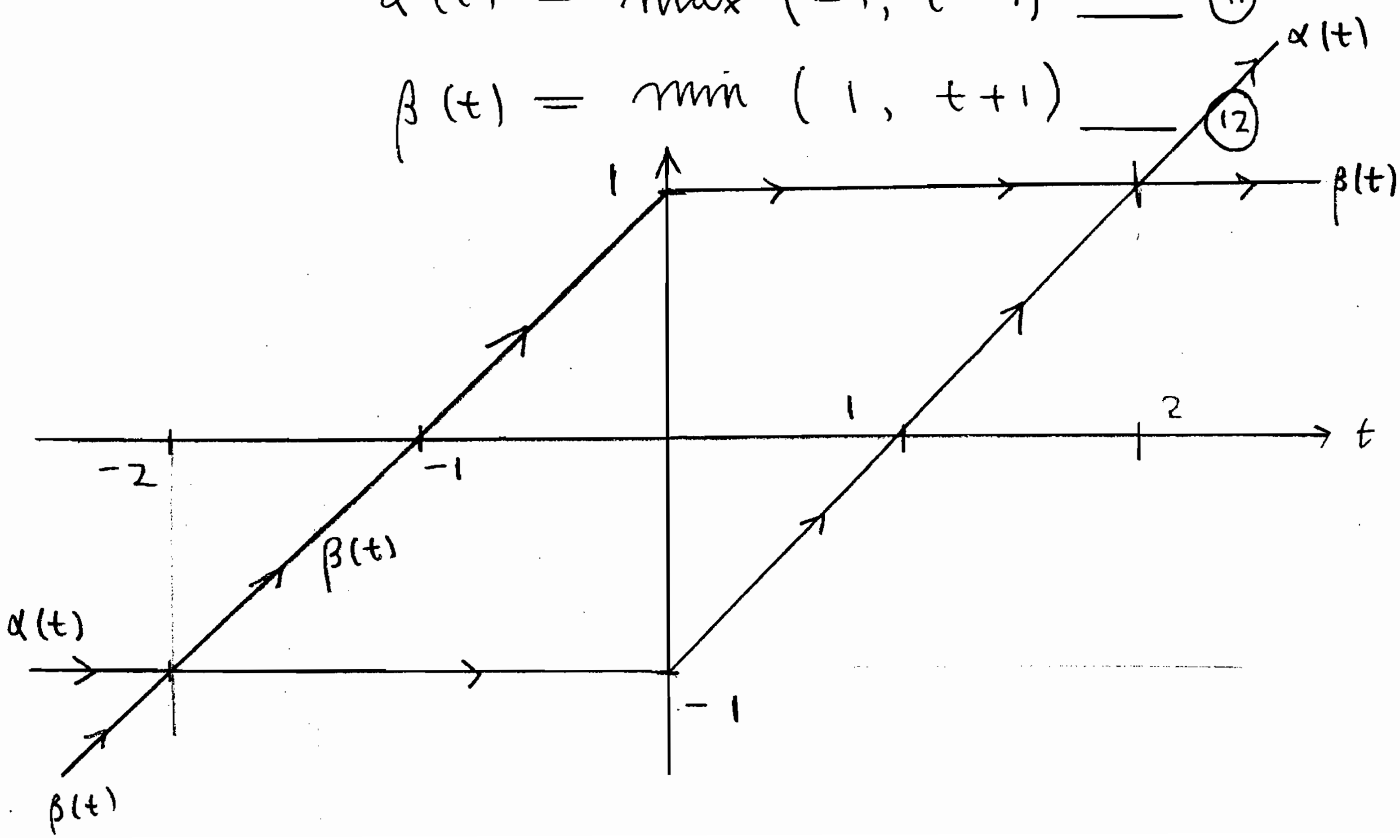
$$y(t) = \int_{-\infty}^{\infty} \underbrace{I[-1,1](\tau)}_{x(\tau)} \; \underbrace{I[t-1, t+1](\tau)}_{h(t-\tau)} \, d\tau$$

$$= \int_{-\infty}^{\infty} I[\alpha(t), \beta(t)](\tau) \, d\tau \quad \text{--- (10)}$$

where

$$\alpha(t) = \max(-1, t-1) \quad \text{--- (11)}$$

$$\beta(t) = \min(1, t+1) \quad \text{--- (12)}$$



From figure, look at following time panels:

<u>$t < -2$</u> here $\beta(t) < \alpha(t)$

i.e. $I[\alpha(t), \beta(t)](\tau) = 0$ all τ

thus from 10

$$y(t) = 0 \quad \text{—— } (13)$$

<u>$-2 \leq t < 0$:</u> here

$$\left.\begin{array}{l} \beta(t) = t+1 \\ \alpha(t) = -1 \\ \alpha(t) < \beta(t) \end{array}\right\} \text{—— } (14)$$

From 14 10

$$y(t) = \int_{\alpha(t)}^{\beta(t)} 1 \, d\tau = \int_{-1}^{t+1} \tau \, d\tau = t+2 \quad \text{—— } (15)$$

<u>$0 \leq t \leq 2$:</u> here

$$\left.\begin{array}{l} \alpha(t) = t-1, \\ \beta(t) = 1, \\ \alpha(t) < \beta(t) \end{array}\right\} \text{—— } (16)$$

From (16) (10)

$$y(t) = \int_{t-1}^{1} 1 . d\tau = 1 - (t-1)$$

$$= 2 - t \quad \text{——— (17)}$$

$\underline{t > 2}$ here $\beta(t) < \alpha(t)$

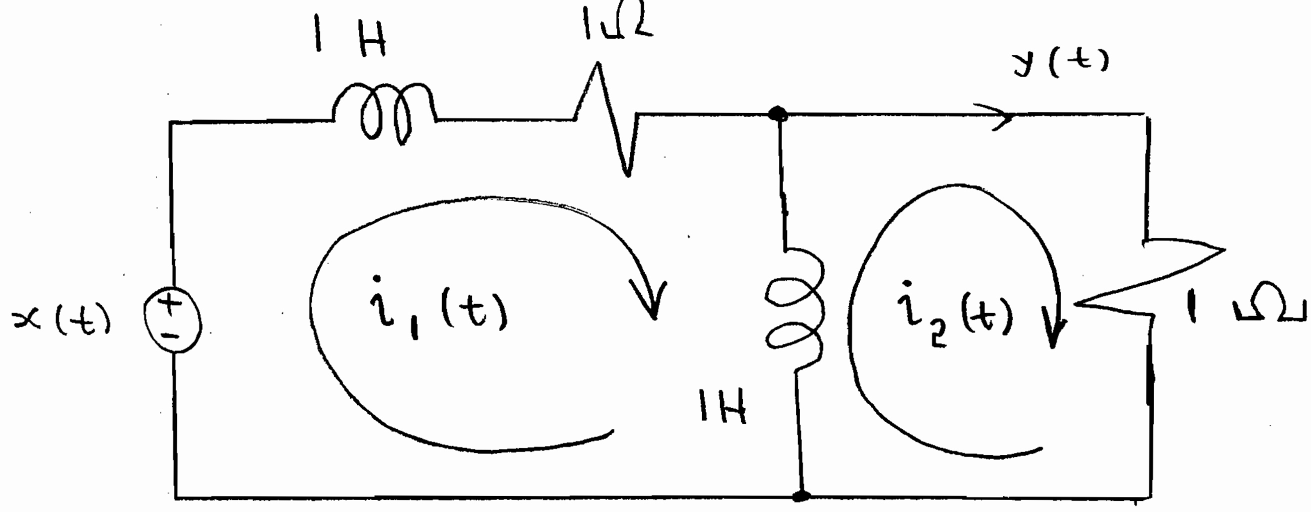
ie $I[\alpha(t), \beta(t)](\tau) = 0$ all τ

ie from (10)

$$y(t) = 0 \quad \text{——— (18)}$$

Collecting (13) (15) (17) (18) find

$$y(t) = \begin{cases} 0, & t < -2 \\ t+2, & -2 \leq t < 0 \\ 2 - t, & 0 \leq t < 2 \\ 0, & t > 2 \end{cases}$$



(i) Introduce the mesh currents i_1 and i_2.

By KVL around path of i_1 get

$$x(t) = (1)\,D i_1(t) + (1)\,i_1(t) + (1)\,D\big(i_1(t) - i_2(t)\big)$$

i.e. $$x(t) = 2Di_1(t) + i_1(t) - Di_2(t)$$

$$x(t) = (2D+1)\,i_1(t) - D\,i_2(t) \quad \text{\_\_\_} ①$$

By KVL around path of i_2 get

$$0 = (1)\,i_2(t) + D\big(i_2(t) - i_1(t)\big)$$

i.e.
$$0 = -Di_1(t) + Di_2(t) + i_2(t)$$

$$0 = -Di_1(t) + (D+1)\,i_2(t) \quad \text{—} (2)$$

By inspection
$$y(t) = i_2(t) \quad \text{—} (3)$$

From (1) and (2)
$$\begin{bmatrix} x(t) \\ 0 \end{bmatrix} = \begin{bmatrix} (2D+1) & -D \\ -D & (D+1) \end{bmatrix} \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} \quad \text{—} (4)$$

From (4) (3) and Cramer rule
$$y(t) = i_2(t) = \frac{\begin{vmatrix} 2D+1 & x(t) \\ -D & 0 \end{vmatrix}}{\begin{vmatrix} 2D+1 & -D \\ -D & D+1 \end{vmatrix}}$$

$$= \frac{0 - (-D)\,x(t)}{(2D+1)(D+1) - (-D)(-D)}$$

i.e.
$$y(t) = \frac{D\,x(t)}{2D^2 + 3D + 1 - D^2}$$

$$= \frac{D\,x(t)}{D^2 + 3D + 1} \quad \text{—— } (5)$$

From (5) get differential equation

$$(D^2 + 3D + 1)\,y(t) = D\,x(t)$$

ie
$$Q(D)\,y(t) = P(D)\,x(t) \quad \text{—— } (6)$$

where
$$Q(D) = D^2 + 3D + 1 \quad \text{—— } (7)$$

$$P(D) = D \quad \text{—— } (8)$$

(ii) (a) $\deg(P) = 1$ $\quad \deg(Q) = 2$

ie $\deg(P) < \deg(Q)$ —— (9)

roots of $Q(\lambda) = \lambda^2 + 3\lambda + 1$

are

$$\left.\begin{aligned} \lambda_1 &= \frac{-3 + \sqrt{9-4}}{2} < 0 \\ \lambda_2 &= \frac{-3 - \sqrt{9-4}}{2} < 0 \end{aligned}\right\} \text{—} (10)$$

From (10) (9) and theorem 2.3.12

system is BIBO - stable.

(b.) Given

$$y(0-) = 1 \qquad y^{(1)}(0-) = 1 \text{—} (11)$$

Take $x(t) = 0$ (the zero signal)

From initial condition (11) see that response

$y(t)$ cannot possibly by the zero signal.

$\therefore$ system <u>NOT</u> linear.

(iii) For $x(t) = u(t)$

have $$Dx(t) = \delta(t)$$

ie $y(t)$ given by

$$(D^2 + 3D + 1)\,y(t) = \delta(t)$$

Thus $\delta(t)$ causes $y(t)$ to be discontinuous at $t = 0$.

3 (a) $$y(t) = 2 + \int_{-\infty}^{\frac{t}{4} - 3} x(\tau)\, d\tau \quad \text{—— ①}$$

Notice that

$$t < \frac{t}{4} - 3$$

iff $$4t < t - 12$$

iff $$3t < -12$$

iff $$t < -4.$$

Thus take e.g. $t_0 = -8$.

$$\frac{t_0}{4} - 3 = -\frac{8}{4} - 3 = -7.$$

i.e. $$\underbrace{\frac{t_0}{4} - 3}_{= -7} > \underbrace{t_0}_{= -8} \quad \text{—— ②}$$

② suggests system not causal.

To show this

Put $x_1(t) = 0$ all t — ③

From ① response to $x_1(t)$ is

$y_1(t) = 2$ for all t — ④

Define

$$x_2(t) = \begin{cases} 0, & t \leq -8 = t_0 \\ 1, & t > -8 = t_0 \end{cases} \quad \text{— ⑤}$$

From ① response to $x_2(t)$ at $t = t_0$ is

$$y_2(t_0) = 2 + \int_{-\infty}^{\frac{t_0}{4} - 3} x_2(\tau)\, d\tau$$

$$= 2 + \int_{-\infty}^{-7} x_2(\tau)\, d\tau$$

$$\overset{⑤}{=} 2 + \int_{-8}^{-7} 1 \, d\tau$$

$$= 1 \quad \text{— ⑥}$$

From 6 4 see

$$y_1(t_0) \neq y_2(t_0).$$

even though (from 3 5)

$$x_1(t) = x_2(t) = 0, \quad t \leqslant t_0$$

i.e. system NOT causal.

(b) $y(t) = x(at)$ ___ 7

Consider the possibilities:

$a = 1$: $y(t) = x(t)$

$\therefore$ system causal. when $a = 1$ ___ 8

$a > 1$: Fix $t_0 = 1$. ___ 9

$\therefore$ $y(t_0) = x(a \cdot t_0)$ ___ 10

Take $t_0 = 1$.

$x_1(t) = 0$ all t. ___ 11

$$x_2(t) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < t_0. \end{cases} \quad \text{——} \; (12)$$

then
$$y_1(t_0) \overset{(10)\,(9)}{=} x_1(a\,t_0) \overset{(11)}{=} 0 \quad \text{——} \; (13)$$

$$y_2(t_0) \underset{(10)\,(9)}{=} x_2(a) = 1 \quad \text{——} \; (14)$$

from (12) since $a > t_0$

From (12) (11) see
$$x_1(t) = x_2(t) \qquad \text{all } t \leq t_0$$

But from (13) (14) $\quad y_1(t_0) \neq y_2(t_0)$.

conclude: System <u>NOT</u> causal when $a > 1$ —— (15)

<u>$0 \leq \alpha < 1$:</u> Fix $t_0 = -1$. —— (16)

then $\quad \alpha\, t_0 > t_0$ —— (17)

Now define $x_1(t)$ by (11)

" $\quad x_2(t)$ by (12)

Then $y_1(t_0) \overset{7}{=} x_1(at_0) \overset{11}{=} 0$

and $y_2(t_0) \overset{7}{=} x_2(at_0) = 1$

from 12 17

Thus $y_1(t_0) \neq y_2(t_0)$

even though

$$x_1(t) = x_2(t), \quad \text{all } t \leq t_0$$

Conclude: system <u>NOT</u> causal when $0 \leq a < 1$.

<u>$a < 0$</u> Fix $t_0 = -1$. ___ 18

Then $at_0 = -a > 0$

since $a < 0$

Thus $at_0 > t_0$.

Now <u>exactly</u> as for the case $0 \leq a < 1$

we conclude

system <u>NOT</u> causal when $a < 0$.

Thus

$$a = 1$$

is only value for which system is causal.

(iii)

$$y(t) = x(t^3) \quad \text{—— (19)}$$

Fix input $x_1(t)$ with response $y_1(t)$.

Thus

$$y_1(t) = x_1(t^3) \quad \text{—— (20)}$$

Put

$$x_2(t) = x_1(t - t_0) \quad \text{—— (21)}$$

for some fixed t_0

From (19) response $y_2(t)$ is

$$y_2(t) = x_2(t^3)$$

$$= x_1(t^3 - t_0) \quad \text{—— (22)}$$

↑
just replace t by t^3 in (21)!

But

$$y_1(t - t_0) = x_1((t - t_0)^3) \quad \text{—— (23)}$$

↑
replace t by $t - t_0$ in (20)

Now generally $\quad t^3 - t_0 \neq (t - t_0)^3$

thus $\quad x_1(t^3 - t_0) \neq x_1(t - t_0)^3$ ___ 24

thus from 24 23 22

$$y_2(t) \neq y_1(t - t_0)$$

conclude :

system <u>NOT</u> time invariant.