UNIVERSITY OF WATERLOO
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
FINAL EXAMINATION

Course Abbreviation and Number: ECE-342
Course Title: Signals and Systems
Sections: 001 (Elec.) and 002 (Comp.)
Instructor: A.J. Heunis

Date of Examination: Friday August 7, 2009
Time Period: Start time 9:00am, End time 11.30am
Duration of exam: 2 hours 30 minutes
Number of exam pages (including cover sheet): 8 pages
Exam Venue: PAC 1 and 2
Special aids permitted: Hand calculators only

Answer all five questions
Total marks = 100
Questions 1, 2, 4, 5 carry 20 marks each, and qu. 3 carries 25 marks (i.e. 5 bonus marks!)
Mark allocation within questions is shown.

Please indicate on your exam book whether you are COMPUTER Engineering (Q) or ELECTRICAL Engineering (E). This is VERY important !!!

1. The zero-state response of a linear time-invariant system, when the input signal is the unit step function, is given by

$$y(t) = 1 - e^{-t} - te^{-t}.$$

(a) [6] Determine the transfer function of the system.
(b) [3] Determine if the system is BIBO-stable.
(c) [6] Determine the steady-state response of the system to the input signal

$$x(t) = \begin{cases} 10\cos[t + \pi/4], & \text{for all } t \geq 0, \\ 0, & \text{for all } t < 0. \end{cases}$$

(d) [5] Let $y(t)$ be the response to the input signal

$$x(t) = 10u(t) + \sum_{n=1}^{10} e^{-nt}.$$

Determine

$$\lim_{t \to \infty} y(t).$$

2(a). [10] Determine the z-transform of the discrete *ramp* signal

$$r[k] = \begin{cases} 0, & \text{for all } k \leq 0, \\ k, & \text{for all } k \geq 0. \end{cases}$$

Then determine the z-transform of the signal

$$x[k] = 2^k r[k].$$

(b) [10] A discrete-time system is linear time-invariant causal with impulse response

$$h[k] = k, \qquad \text{for all } k \geq 0.$$

An input signal $x[k]$ is right-sided, and the corresponding output signal is

$$y[k] = k^2, \qquad \text{for all } k \geq 0.$$

Determine

(i) [3] Show that $y[k] = 0$ for all $k < 0$;
(ii) [7] Determine $x[k]$ for all $k \geq 0$.

Continues next page ......

3(a). [10] Determine the average power in the following signal:

$$x(t) = \cos^2(20t)\sin(10t), \qquad -\infty < t < \infty.$$

(b) Using the result from (a), or otherwise, determine the average power in the following signals:

(i) [10] $y(t) = \cos^2(20t)\sin(10t)e^{-20jt}, \qquad -\infty < t < \infty;$

(ii) [5] $z(t) = \cos^2(40t)\sin(20t), \qquad -\infty < t < \infty.$

4(a). [10] Determine the Fourier transform of the signal

$$\frac{1}{1 - jt}, \qquad -\infty < t < \infty.$$

(b) [10] You are given that the Fourier transform of the signal

$$x(t) = \exp\left(-\frac{t^2}{2}\right) \qquad \text{is} \qquad X(\omega) = \sqrt{2\pi}\exp\left(-\frac{\omega^2}{2}\right).$$

Use this to determine the Fourier transform of the signal

$$y(t) = \exp\left(-\frac{t^2}{4}\right).$$

Then determine the signal $z(t)$ which satisfies the convolution equation

$$(z * x)(t) = y(t).$$

5(a). [7] Determine the Fourier transform of the "signum" function $\text{sgn}(t)$, which is given by

$$\text{sgn}(t) = \begin{cases} 1, & \text{for all } t > 0, \\ 0, & \text{for } t = 0, \\ -1, & \text{for all } t < 0. \end{cases}$$

Hint: Write $\text{sgn}(t)$ in terms of $u(t)$.

(b) [6] Determine the Fourier transform

$$\mathcal{F}\left\{\frac{2}{jt}\right\}(\omega).$$

(c) [7] Determine the Fourier transform

$$\mathcal{F}\left\{\frac{1}{t^2}\right\}(\omega).$$

USEFUL FACTS:

Trig. formulae: For α in radians we have

$$\cos\alpha = \frac{e^{j\alpha}+e^{-j\alpha}}{2}, \qquad \sin\alpha = \frac{e^{j\alpha}-e^{-j\alpha}}{2j}.$$

Energy in signals:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt, \qquad E = \sum_{k=-\infty}^{\infty} |x[k]|^2.$$

Average power in signals:

$$P = \lim_{a\to\infty} \frac{1}{2a}\int_{-a}^{a} |x(t)|^2 dt, \qquad P = \lim_{n\to\infty} \frac{1}{2n+1}\sum_{k=-n}^{n} |x[k]|^2.$$

Even and odd parts of a signal:

$$x_e(t) = \frac{1}{2}[x(t)+x(-t)], \qquad x_o(t) = \frac{1}{2}[x(t)-x(-t)],$$

Sifting formulae: For a continuous-time signal $x(t)$ or discrete-time signal $x[k]$ we have

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad \text{(continuous)}, \qquad x[k] = \sum_{n=-\infty}^{\infty} x[n]\delta[k-n] \quad \text{(discrete)}.$$

For a continuous-time signal $x(t)$, for which $x(t)$, $x^{(1)}(t)$, $x^{(2)}(t), \ldots, x^{(n)}(t)$, are continuous functions of t, we have

$$x^{(n)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(n)}(t-\tau)d\tau, \qquad n = 1, 2, \ldots$$

Convolution of two signals:

$$(x_1 * x_2)(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau, \quad \text{(continuous)} \qquad (x_1 * x_2)[k] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[k-n], \quad \text{(discrete)}.$$

Output of a linear system: A continuous-time linear time-invariant system with impulse response $h(t)$ [a discrete-time linear time-invariant system with impulse response $h[k]$] and initially at rest, has response to a continuous-time input $x(t)$ [discrete-time input $x[k]$] given by

$$y(t) = (h*x)(t) \quad \text{in continuous-time}, \qquad y[k] = (h*x)[k] \quad \text{in discrete-time}.$$

Zero-input response: For the linear system

$$(*)\begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

with

$$Q(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\dots(\lambda - \lambda_r)^{m_r},$$

the zero-input response is given by

$$y_{zi}(t) \triangleq \sum_{i=1}^{r}\sum_{j=1}^{m_i} c_{i,j}t^{j-1}e^{\lambda_i t},$$

where constants $c_{i,j}$ are determined from the initial conditions.

Heaviside expansion theorem: Let $X(s) \triangleq N(s)/D(s)$ be a coprime rational function such that $\deg(N) < \deg(D)$, and factorize the denominator polynomial $D(s)$ as follows

$$D(s) = a_n(s - p_1)^{m_1}(s - p_2)^{m_2}\dots(s - p_l)^{m_l}.$$

Then $X(s)$ can be expanded as

$$X(s) = \sum_{i=1}^{l}\left\{\sum_{j=1}^{m_i}\frac{r_{ij}}{(s-p_i)^j}\right\}, \quad \text{where} \quad r_{ij} = \frac{1}{(m_i - j)!}\frac{d^{m_i-j}}{ds^{m_i-j}}\left[X(s)(s-p_i)^{m_i}\right]\big|_{s=p_i},$$

for $i = 1, 2, \dots, l$, $j = 1, 2, \dots, m_i$.

Table of Laplace Transforms: For a continuous-time signal $x(t)$, $-\infty < t < \infty$ define

$$\mathcal{L}\{x(t)\}(s) \equiv X(s) \triangleq \int_{0-}^{\infty} x(t)e^{-st}\,dt.$$

Signal $x(t)$, $t \geq 0$	Laplace Transform $X(s)$
$\delta(t)$	1
$\delta^{(n)}(t)$, $n = 1, 2, \dots$	s^n
$u(t)$	$1/s$
t^n, $n = 0, 1, 2, \dots$	$n!/s^{n+1}$
$e^{\alpha t}$, α complex	$1/(s - \alpha)$
$t^n e^{\alpha t}$, α complex $n = 0, 1, 2, \dots$	$n!/(s-\alpha)^{n+1}$
$\sin\omega t$	$\omega/(s^2 + \omega^2)$
$\cos\omega t$	$s/(s^2 + \omega^2)$
$e^{\alpha t}\sin\omega t$	$\omega/[(s-\alpha)^2 + \omega^2]$
$e^{\alpha t}\cos\omega t$	$(s-\alpha)/[(s-\alpha)^2 + \omega^2]$

Main Properties of Laplace Transforms: Suppose signals $x(t)$ and $y(t)$ have Laplace transforms $X(s)$ and $Y(s)$ respectively, and let α be a complex constant. Then:

$$\begin{aligned}
\mathcal{L}\{x^{(n)}(t)\}(s) &= s^nX(s) - s^{n-1}x(0-) - \dots - sx^{(n-2)}(0-) - x^{(n-1)}(0-). \\
\mathcal{L}\left\{\int_{0-}^{t} x(\tau)\,d\tau\right\}(s) &= \frac{X(s)}{s}.
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}\{e^{\alpha t}x(t)\}(s) &= X(s-\alpha). \\
\mathcal{L}\{(x*y)(t)\}(s) &= X(s)Y(s). \\
\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s). \\
x(0) &= \lim_{s\to\infty} sX(s).
\end{aligned}
$$

Frequency Response: Suppose the system (*) is BIBO stable with transfer function $H(s) \triangleq P(s)/Q(s)$. If the input signal $x(t)$ is given by $x(t) = A\cos(\omega t + \theta)$, $t \geq 0$, $x(t) = 0$, $t < 0$, with corresponding output $y(t)$ then

$$\lim_{t\to\infty} y(t) = y_{ss}(t) \quad \text{where} \quad y_{ss}(t) = A|H(j\omega)|\cos(\omega t + \theta + \angle H(j\omega)).$$

Table of z-Transforms: For a discrete-time signal $x[k]$, $-\infty < k < +\infty$ define

$$\mathcal{Z}\{x[k]\}(z) \equiv X(z) \triangleq \sum_{k=0}^{\infty} x[k]z^{-k}.$$

Signal $x[k]$, $k \geq 0$	z-transform $X(z)$
$\delta[k]$	1
$\delta_n[k]$, $n = 1, 2, \ldots$	z^{-N}
$u[k]$	$z/(z-1)$
α^k, α complex	$z/(z-\alpha)$
$(k)_{n-1}\alpha^{k-n+1}/(n-1)!$ α complex, $n = 0, 1, 2, \ldots$	$z/(z-\alpha)^n$
$\sin(k\omega T)$	$[z\sin(\omega T)]/[z^2 - 2z\cos(\omega T) + 1]$
$\cos(k\omega T)$	$[z^2 - z\cos(\omega T)]/[z^2 - 2z\cos(\omega T) + 1]$
$\alpha^k \sin(k\omega T)$	$[\alpha z\sin(\omega T)]/[z^2 - 2\alpha z\cos(\omega T) + \alpha^2]$
$\alpha^k \cos(k\omega T)$	$[z^2 - \alpha z\cos(\omega T)]/[z^2 - 2\alpha z\cos(\omega T) + \alpha^2]$

Main Properties of z-Transforms: Let signals $x[k]$ and $y[k]$ have z-transforms $X(z)$ and $Y(z)$ respectively, let α be a complex constant, and let N be a positive integer. Then:

$$
\begin{aligned}
\mathcal{Z}\{x[k+N]\}(z) &= z^N X(z) - z^N x[0] - z^{N-1}x[1] - \ldots - z^2 x[N-2] - zx[N-1]. \\
\mathcal{Z}\{x[k-N]\}(z) &= z^{-N}X(z) + \left\{x[-N] + z^{-1}x[1-N] + z^{-2}x[2-N] + \ldots + z^{1-N}x[-1]\right\}. \\
\mathcal{Z}\{\alpha^k x[k]\}(z) &= X\left(\frac{z}{\alpha}\right). \\
\mathcal{Z}\{x*y\}(z) &= X(z)Y(z). \\
\lim_{k\to\infty} x[k] &= \lim_{z\to 1}(z-1)X(z). \\
x[0] &= \lim_{z\to\infty} X(z). \\
\mathcal{Z}\{kx[k]\}(z) &= -z\frac{dX(z)}{dz}.
\end{aligned}
$$

Fourier Series: Let $x(t)$ be a periodic signal with a period $T > 0$. The exponential Fourier series expansion of $x(t)$ is

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \text{where} \quad \omega_0 = \frac{2\pi}{T}, \quad \text{and} \quad a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\, dt.$$

Parseval Theorem for Fourier Series: Suppose that $x(t)$ is a periodic signal satisfying the Dirichlet conditions. Then the average power in $x(t)$ is given by

$$P = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

Fourier Transform Relations: Suppose $x(t)$ is a signal and define

$$X(\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt. \quad \text{Then} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega,$$

provided the integrals make sense.

Main Properties of Fourier Transforms: Suppose that $x(t)$ and $y(t)$ are signals with Fourier transforms $X(\omega)$ and $Y(\omega)$ respectively and let t_0, ω_0 and a be real constants. Then:

$$\begin{aligned}
X(-\omega) &= X^*(\omega), \qquad \text{when } x(t) \text{ is real-valued.} \\
\mathcal{F}\{x(t-t_0)\}(\omega) &= e^{-j\omega t_0}X(\omega). \\
\mathcal{F}\{e^{j\omega_0 t}x(t)\} &= X(\omega-\omega_0). \\
\mathcal{F}\{x^{(n)}(t)\}(\omega) &= (j\omega)^n X(\omega), \qquad n = 1, 2, 3, \ldots \\
\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\}(\omega) &= \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \\
\mathcal{F}\{x(at)\}(\omega) &= \frac{1}{|a|}X\left(\frac{\omega}{a}\right). \\
\mathcal{F}\{X(t)\}(\omega) &= 2\pi x(-\omega). \\
\mathcal{F}\{x(t)*y(t)\}(\omega) &= X(\omega)Y(\omega). \\
\mathcal{F}\{x(t)y(t)\}(\omega) &= \frac{1}{2\pi}(X*Y)(\omega). \\
\mathcal{F}\{x(-t)\}(\omega) &= X(-\omega).
\end{aligned}$$

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Table of Fourier Transforms:

Signal $x(t)$	Fourier Transform $X(\omega)$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$\delta^{(n)}(t)$, $n = 1, 2, \ldots$	$(j\omega)^n$
$u(t)$	$1/(j\omega) + \pi\delta(\omega)$
$e^{-\alpha t}u(t)$, α complex, $\mathrm{re}(\alpha) > 0$,	$1/(\alpha + j\omega)$
$t^n e^{-\alpha t}u(t)$, α complex, $\mathrm{re}(\alpha) > 0$, $n = 0, 1, 2, \ldots$	$n!/(\alpha + j\omega)^{n+1}$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$(\pi/j)[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$

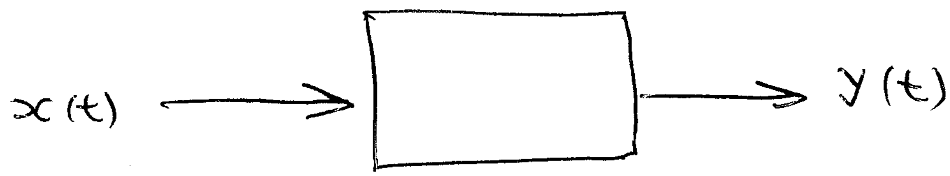
Parseval Theorem for Fourier Transforms: Suppose that signal $x(t)$ has Fourier transform $X(\omega)$ and is such that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty.$$

Then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.$$

1.



unit step response is

$$y(t) = 1 - e^{-t} - t e^{-t}.$$

(i) Thus transfer function is

$$H(s) \triangleq \frac{\mathcal{L}\{y(t)\}(s)}{\mathcal{L}\{x(t)\}(s)}$$

for $x(t) \triangleq u(t)$.

$$\therefore \quad \mathcal{L}\{x(t)\}(s) = \frac{1}{s}$$

and

$$\mathcal{L}\{y(t)\}(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$= \frac{(s+1)^2 - s(s+1) - s}{s(s+1)^2}$$

$$= \frac{[s^2 + 2s + 1] - s^2 - s - s}{s(s+1)^2}$$

$$\therefore \mathcal{L}\{y(t)\} = \frac{1}{s(s+1)^2}$$

$$\therefore H(s) = \frac{\mathcal{L}\{y(t)\}(s)}{\mathcal{L}\{x(t)\}(s)}$$

$$= \frac{1}{(s+1)^2}$$

(ii) $Q(s) = (s+1)^2$.

Hence the roots of $Q(s)$ are -1

i.e. system is BIBO - stable.

(iii) Let $y(t)$ be response to input signal

$$x(t) = 10 \cos[t + \pi/4], \quad t \geq 0.$$

Then

$$y(t) \longrightarrow y_{ss}(t)$$

where

$$y_{ss}(t) = 10\,|H(j\omega)| \cos[t + \pi/4 + \underline{/H(j\omega)}]$$

and $\omega = 1$ rad/sec.

$$\therefore \quad H(j\omega) = H(j)$$

$$= \frac{1}{(1+j)^2}$$

$$= \frac{1}{(\sqrt{2} \angle \pi/4)^2}$$

$$= \frac{1}{2} \angle -\pi/2$$

$$\therefore \quad |H(j\omega)| = \frac{1}{2}, \quad \angle H(j\omega) = -\frac{\pi}{2}$$

Whence

$$y_{ss}(t) = 5 \cos\left[t + \pi/4 - \pi/2\right]$$

$$= 5 \cos\left[t - \frac{\pi}{4}\right].$$

(iv) Let $y(t)$ be response to input signal

$$x(t) = 10\, u(t) + \sum_{n=1}^{10} e^{-nt}$$

By linearity we have

$$y(t) = 10\,y_0(t) + \sum_{1}^{10} y_n(t)$$

where $y_0(t)$ is responce to unit-step input namely

$$y_0(t) = 1 - e^{-t} - t\,e^{-t}$$

and $y_n(t)$ is response to the input

$$x_n(t) = e^{-nt}.$$

$$\therefore \quad Y_n(s) = H(s).\,X_n(s)$$

$$= \frac{1}{(s+1)^2} \cdot \frac{1}{(s+n)}$$

By Final Value Theorem

$$\lim_{t \to \infty} y_n(t) = \lim_{s \to 0} s\,Y_n(s)$$

$$= \lim_{s \to 0} \frac{s}{(s+1)^2(s+n)}$$

$$= 0.$$

also clearly

$$\lim_{t \to \infty} y_0(t) = 1,$$

hence

$$\lim_{t \to \infty} y(t) = 10.$$

2 (a) $$r[k] = k \cdot u[k]$$

Then

$$\mathcal{Z}\{r[k]\}(z) = \mathcal{Z}\{k \cdot u[k]\}(z)$$

$$= -z \frac{d}{dz} \mathcal{Z}\{u[k]\}(z) \quad \text{—} \; 1$$

But from tables

$$\mathcal{Z}\{u[k]\}(z) = \frac{z}{z-1} \quad \text{—} \; 2$$

Then from 1 2

$$\mathcal{Z}\{r[k]\}(z) = -z \frac{d}{dz} \left[\frac{z}{z-1} \right]$$

$$= -z \left[\frac{(1)(z-1) - z(1)}{(z-1)^2} \right]$$

$$= \frac{z}{(z-1)^2}$$

2(a) continued

Let $$x[k] = 2^k \, r[k].$$

Then

$$\mathcal{Z}\{x[k]\}(z) = \mathcal{Z}\{2^k \, r[k]\}(z)$$

$$= \mathcal{Z}\{r[k]\}\left(\frac{z}{2}\right)$$

(by rescaling property)

$$= \frac{(z/2)}{\left[\frac{z}{2} - 1\right]^2}$$

$$= \frac{2z}{(z-2)^2}$$

(b) system LTI and causal with

$$h[k] = k, \quad k \geq 0.$$

By causality

$$h[k] = 0, \quad \text{all } k < 0$$

ie

$$h[k] = k \cdot u[k] \quad \text{---} \quad (1)$$

Input signal is right-sided, thus

$$x[k] = 0, \quad \text{all } k < 0 \quad \text{---} \quad (2)$$

and corresponding output signal is

$$y[k] = k^2, \quad \text{all } k \geq 0 \quad \text{---} \quad (3)$$

Now

$$y[k] = (h * x)[k]$$

$$= \sum_{n=-\infty}^{\infty} h[n]\, x[k-n]$$

$$\stackrel{(1)}{=} \sum_{n=0}^{\infty} h[n]\, x[k-n] \quad \text{— (4)}$$

(i) when $k < 0$: then

$$k - n < 0 \quad \text{for all } n \geq 0$$

i.e. from (2)

$$x[k-n] = 0 \quad \text{"} \quad \text{"} \quad \text{— (5)}$$

From (4) (5)

$$y[k] = 0. \quad \text{— (6)}$$

(ii) From (6) and (3)

$$y[k] = k^2 . u[k] \quad \text{— (7)}$$

By convolution property of z - transforms

$$\left.\begin{aligned} Y(z) &= H(z)\,X(z) \\ \text{where} \quad H(z) &\triangleq \mathcal{Z}\{h[k]\}(z) \end{aligned}\right\} \text{— (8)}$$

From (1) and tables

$$H(z) = \frac{z}{(z-1)^2} \quad \text{——} \quad 9$$

From 7 and tables

$$Y(z) = \frac{z(z+1)}{(z-1)^3} \quad \text{——} \quad 10$$

By 8 to 10 :

$$\frac{z(z+1)}{(z-1)^3} = \frac{z}{(z-1)} X(z)$$

then

$$X(z) = \frac{z+1}{z-1} \quad \text{——} \quad 11$$

By partial fractions

$$\frac{X(z)}{z} = \frac{z+1}{z(z-1)}$$

$$= \frac{2}{z-1} - \frac{1}{z}$$

thus

$$X(z) = \frac{2z}{z-1} - 1$$

Then inverse z - transforms give

$$x[k] = 2u[k] - \delta[k]$$

3 (a)

$$x(t) = \cos^2(20t)\sin(10t) \quad \text{———} \quad (1)$$

Easiest to use Parseval: Determine exp. Fourier coefficients first. From (1)

$$x(t) = \left[\frac{e^{j20t} + e^{-j20t}}{2}\right]^2 \left[\frac{e^{j10t} - e^{-j10t}}{2j}\right]$$

$$= \frac{1}{8j}\left[e^{j40t} + 2 + e^{-j40t}\right]\left[e^{j10t} - e^{-j10t}\right]$$

$$= \frac{1}{8j}\left[e^{j50t} - e^{j30t} + 2e^{j10t} - 2e^{-j10t} + e^{-j30t} - e^{-j50t}\right]$$

$$= \left(\frac{1}{8j}\right) e^{j50t} - \left(\frac{1}{8j}\right) e^{j30t}$$

$$+ \frac{1}{4j} e^{10jt} - \frac{1}{4j} e^{-10jt}$$

$$+ \frac{1}{8j} e^{-j30t} - \frac{1}{8j} e^{-j50t} \quad \text{— (2)}$$

From (2) have

$$\omega_0 = 10$$

with exponential coeffts:

$$a_5 = \frac{1}{8j} \qquad a_{-5} = \frac{-1}{8j}$$

$$a_3 = -\frac{1}{8j} \qquad a_{-3} = \frac{1}{8j}$$

$$a_1 = \frac{1}{4j} \qquad a_{-1} = \frac{-1}{4j}$$

By Parseval: average power of $x(t)$ is

$$P_x = \frac{1}{8^2} + \frac{1}{8^2} + \frac{1}{8^2} + \frac{1}{8^2} + \frac{1}{4^2} + \frac{1}{4^2}$$

$$= \frac{1}{64}\left[4 + 4 + 4\right]$$

$$= \frac{3}{16}.$$

(b) (i)

$$y(t) = \overbrace{\cos^2(20t)\sin(10)}^{x(t)}\, e^{-20jt} \quad \text{—} \quad 3$$

From 2 see that $y(t)$ is periodic with

$$\omega_0 = 10.$$

Now

$$|y(t)|^2 \overset{3}{=} |x(t)e^{-j20t}|^2$$

$$= |x(t)|^2 \underbrace{|e^{-20jt}|^2}_{=1}$$

$$= |x(t)|^2 \quad \text{—— } (4)$$

Average power of $y(t)$ is

$$P_y = \frac{1}{T}\int_0^T |y(t)|^2\, dt$$

$$\overset{(4)}{=} \frac{1}{T}\int_0^T |x(t)|^2\, dt$$

$$= P_x = \frac{3}{16}$$

(ii) $$z(t) = \cos^2(40t)\sin(20t) \quad \text{—— } (5)$$

From (5) and (1)

$$z(t) = x(2t) \quad \text{—— } (6)$$

Now $x(t)$ has period

$$T = \frac{2\pi}{\omega_0} = \frac{\pi}{5}$$

then

$$z(t+T) \overset{6}{=} x(2(t+T))$$

$$= x(2t + 2T)$$

$$= x(2t)$$

(since $x(t)$ has period $2T$)

$$= z(t)$$

ie $z(t)$ also has period T —— 7

average power in z is

$$P_z \overset{7}{=} \frac{1}{T}\int_0^T |z(t)|^2 \, dt$$

$$\overset{⑥}{=} \frac{1}{T}\int_0^T |x(2t)|^2\,dt$$

$$= \frac{1}{T}\int_0^{2T} |x(\tau)|^2 \overbrace{\left(\frac{1}{2}\,d\tau\right)}^{dt} \qquad \tau = 2t$$

$$= \frac{1}{2T}\int_0^{2T} |x(\tau)|^2\,d\tau$$

$$= \frac{1}{2}\left[\frac{1}{T}\int_0^T |x(\tau)|^2\,d\tau + \frac{1}{T}\int_T^{2T} |x(\tau)|^2\,d\tau\right]$$

$$= \frac{1}{2}\left[\frac{1}{T}\int_0^T |x(\tau)|^2\,d\tau + \frac{1}{T}\int_0^T |x(\tau)|^2\,d\tau\right]$$

$$= \frac{1}{2}\left(P_x + P_x\right)$$

$$= P_x = \frac{3}{16}.$$

4. (a) $$y(t) = \frac{1}{1-jt} \quad \text{——} \ (1)$$

From tables

$$\mathcal{F}\{e^{-t}u(t)\}(\omega) = \frac{1}{1+j\omega} \quad \text{——} \ (2)$$

From time-scaling, for $a \neq 0$

$$\mathcal{F}\{e^{-at}u(at)\}(\omega) \overset{(2)}{=} \frac{1}{|a|} \cdot \frac{1}{1+j(\omega/a)} \quad \text{——} \ (3)$$

Take $a = -1$ in (3):

$$\mathcal{F}\{e^{t}u(-t)\}(\omega) = \frac{1}{1-j\omega} \quad \text{——} \ (4)$$

Put $$y_1(t) \triangleq e^{t}u(-t) \quad \text{——} \ (5)$$

From (4) $$Y_1(\omega) = \frac{1}{1-j\omega} \quad \text{——} \ (6)$$

In view of (6) (1) have

$$y(t) = Y_1(t) \quad \text{—— (7)}$$

Then

$$\mathcal{F}\{y(t)\}(\omega) = \mathcal{F}\{Y_1(t)\}(\omega)$$

$$\underset{\substack{\uparrow \\ \text{by duality!}}}{=} 2\pi\, y_1(-\omega)$$

$$\overset{(5)}{=} 2\pi e^{-\omega} u(\omega)$$

4(b).

$$x(t) = \exp\left(-\frac{t^2}{2}\right) \quad \text{———} \; 1$$

has Fourier transform

$$X(\omega) = \sqrt{2\pi} \cdot \exp\left(-\frac{\omega^2}{2}\right) \quad \text{———} \; 2$$

Put

$$y(t) = \exp\left(-\frac{t^2}{4}\right) \quad \text{———} \; 3$$

From 1 3

$$y(t) = x\left(\frac{t}{\sqrt{2}}\right)$$

$$= x(at) \quad \text{for } a = \frac{1}{\sqrt{2}} \quad \text{———} \; 4$$

Then

$$Y(\omega) = \mathcal{F}\{y(t)\}(\omega)$$

$$\overset{4}{=} \mathcal{F}\{x(at)\}(\omega)$$

$$\underset{\substack{\uparrow \\ \text{rescaling}}}{=} \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

$$= \sqrt{2} \,.\, X(\omega\sqrt{2})$$

$$\overset{(2)}{=} \sqrt{2} \,.\, \sqrt{2\pi} \, \exp\left[-\frac{2\omega^2}{2} \right]$$

$$= 2\sqrt{\pi} \,.\, \exp[-\omega^2] \quad \_\_ \; (5)$$

Determine z such that

$$(z * x)(t) = y(t) \quad \_\_ \; (6)$$

Take Fourier transforms on each side of (6).

By convolution property

$$Z(\omega) \,.\, X(\omega) = Y(\omega) \quad \_\_ \; (7)$$

Now combine (7) (5) (2).

$$Z(\omega) \,.\, \sqrt{2\pi} \, \exp\left[-\frac{\omega^2}{2} \right]$$

$$= 2\sqrt{\pi} \, \exp[-\omega^2]$$

thus

$$Z(\omega) = \sqrt{2}\,\exp\left[-\frac{\omega^2}{2}\right] \quad \text{— (8)}$$

From (8)

$$\sqrt{\pi}\,Z(\omega) = \sqrt{2\pi}\,\exp\left[-\frac{\omega^2}{2}\right]$$

$$\overset{(2)}{=} X(\omega)$$

ie

$$Z(\omega) = \frac{1}{\sqrt{\pi}}\,X(\omega) \quad \text{— (9)}$$

Then

$$z(t) = \mathcal{F}^{-1}\{Z(\omega)\}(t)$$

$$= \frac{1}{\sqrt{\pi}}\,\mathcal{F}^{-1}\{X(\omega)\}(t)$$

$$= \frac{1}{\sqrt{\pi}}\cdot\exp\left(\frac{-t^2}{2}\right)$$

5 (a) $$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \quad \text{—— ①}$$

then

$$\mathrm{sgn}(t) = u(t) - u(-t)$$

ie $$\mathcal{F}\{\mathrm{sgn}(t)\}(\omega)$$

$$= \mathcal{F}\{u(t)\}(\omega) - \mathcal{F}\{u(-t)\}(\omega) \quad \text{—— ②}$$

From tables

$$\mathcal{F}\{u(t)\}(\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \quad \text{—— ③}$$

also

$$\mathcal{F}\{u(-t)\}(\omega) = \mathcal{F}\{u(at)\}(\omega)$$

for $a = -1$

re-scaling! $$\overset{\downarrow}{=} \frac{1}{|a|} \mathcal{F}\{u(t)\}\left(\frac{\omega}{a}\right)$$

$$= \mathcal{F}\{u(t)\}(-\omega)$$

$$\overset{3}{=} \frac{1}{j(-\omega)} + \pi\delta(-\omega)$$

$$\underset{\substack{\uparrow \\ \delta(\cdot)\text{ even}}}{=} -\frac{1}{j\omega} + \pi\delta(\omega) \quad \text{—— } 4$$

From 4 3 2

$$\mathcal{F}\{\operatorname{sgn}(t)\}(\omega) = \frac{2}{j\omega} \quad \text{—— } 5$$

(b) Put

$$x(t) = \operatorname{sgn}(t) \quad \text{—— } 6$$

From (a)

$$X(\omega) = \frac{2}{j\omega} \quad \text{—— } 7$$

Then

$$\mathcal{F}\left\{\frac{2}{jt}\right\} \overset{7}{=} \mathcal{F}\{X(t)\}(\omega)$$

$$\underset{\substack{\uparrow \\ \text{Duality}}}{=} 2\pi x(-\omega)$$

$$\overset{(6)}{=} 2\pi \,\mathrm{sgn}(-\omega)$$

$$= -2\pi \,\mathrm{sgn}(\omega) \quad \text{——} \quad (8)$$

(since sgn (·) is odd function)

(c) Put $y(t) = \frac{1}{t}$ —— (9)

then $y^{(1)}(t) = \frac{0(1) - 1}{t^2}$

$$= \frac{-1}{t^2}$$

i.e. $\frac{1}{t^2} = -y^{(1)}(t)$ —— (10)

then $\mathcal{F}\left\{\frac{1}{t^2}\right\}(\omega) \overset{(10)}{=} \mathcal{F}\left\{-y^{(1)}(t)\right\}(\omega)$

$$= -\mathcal{F}\left\{y^{(1)}(t)\right\}(\omega)$$

$$= -j\omega \, \mathcal{F}\{y(t)\}$$

(by differentiation property, since

$$y(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \pm\infty).$$

$$\overset{9}{=} -j\omega \, \mathcal{F}\left\{\frac{1}{t}\right\}(\omega) \quad \text{—— } 11$$

Now

$$\mathcal{F}\left\{\frac{2}{jt}\right\}(\omega) = \frac{2}{j} \mathcal{F}\left\{\frac{1}{t}\right\}(\omega) \quad \text{—— } 12$$

ie from 12 and 8

$$-2\pi \, \mathrm{sgn}(\omega) = \frac{2}{j} \mathcal{F}\left\{\frac{1}{t}\right\}(\omega)$$

ie.

$$\mathcal{F}\left\{\frac{1}{t}\right\}(\omega) = -j\pi \, \mathrm{sgn}(\omega) \quad \text{—— } 13$$

From 13 and 11

$$\mathcal{F}\left\{\frac{1}{t^2}\right\}(\omega) = -\pi\omega \, \mathrm{sgn}(\omega).$$