Chapter 2

Time Domain Analysis of Continuous Time Linear Systems

The goal of this chapter is to study continuous time linear systems from the *time domain* point of view. The main ideas of this chapter will be the notions of impulse function, impulse response, systems described by linear differential equations with constant coefficients, and the zero-input and zero-state response. In the next chapter we shall extend the main ideas of this chapter by looking at systems from a *frequency domain* point of view.

2.1 The Impulse Function

Informally an impulse function is a mathematical description of a physical situation in which some action of essentially "infinite" intensity occurs over an effectively "infinitesimal" duration. Examples are the force applied to a golfball at the instant it is struck by a club, or the voltage spike occurring in a lightning conductor at the instant of a lightning strike. Impulse functions are an indispensable tool for studying the characteristics of linear systems.

2.1.1 Definition of Impulse Functions

To introduce the notion of an impulse function fix a small number $\epsilon > 0$ and visualize the function $u_\epsilon(t)$ shown in Fig. 2.1:

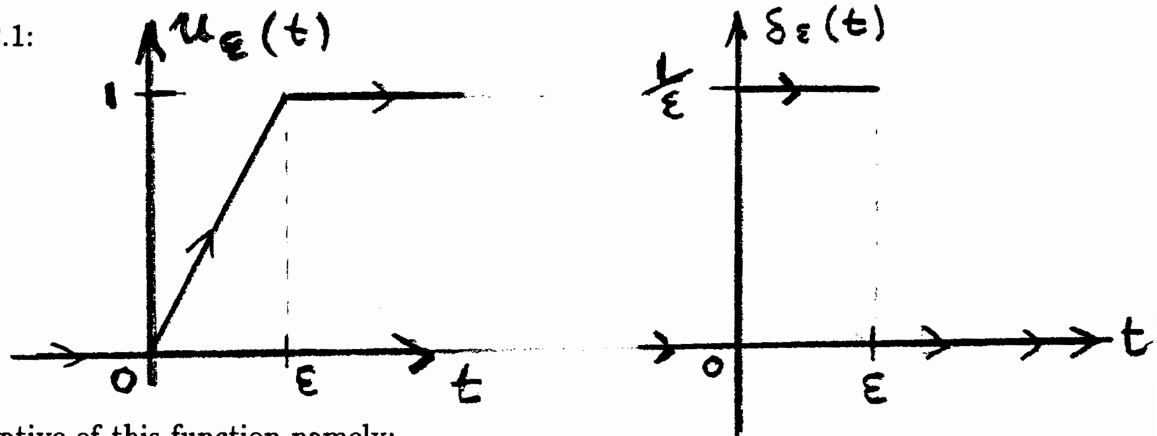


Fig. 2.1.

Let $\delta_\epsilon(t)$ be the derivative of this function namely:

(2.1) $$\delta_\epsilon(t) = \frac{du_\epsilon(t)}{dt},$$

or equivalently,

(2.2) $$u_\epsilon(t) = \int_{-\infty}^{t} \delta_\epsilon(\tau) d\tau.$$

Put

(2.3) $$\delta(t) \triangleq \lim_{\epsilon \to 0} \delta_\epsilon(t).$$

Then, from (2.2), we expect

(2.4) $$u(t) = \lim_{\epsilon \to 0} u_\epsilon(t),$$

thus, taking $\epsilon \to 0$ in (2.2) and using (2.3) and (2.4) gives

(2.5) $$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau,$$

or, equivalently,

(2.6) $$\delta(t) = \frac{du(t)}{dt}.$$

This demonstrates that, at least formally, the function $\delta(t)$ is the derivative of the unit step function. Taking $t \to \infty$ in (2.6) also shows that

(2.7) $$\int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1,$$

and from (2.3) we see that

(2.8) $$\delta(t) = 0 \qquad \text{when } t \neq 0.$$

We call the function $\delta(t)$ given by (2.3) the **impulse function**. The preceding relations show that the impulse function $\delta(t)$ has unit area under its graph *all of which is concentrated at the point* $t = 0$. Thus the impulse function is a somewhat peculiar function whose graph we cannot readily draw but which we represent schematically as an upwards "spike" (see Fig. 2.2).

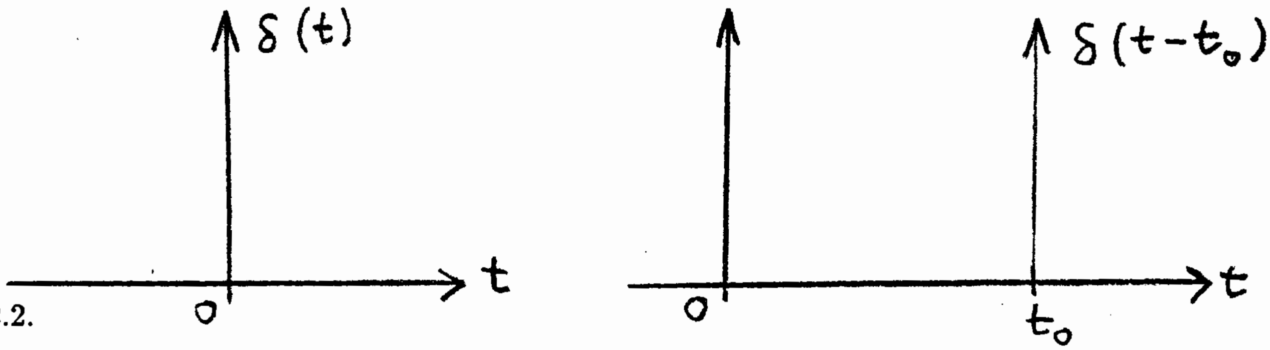


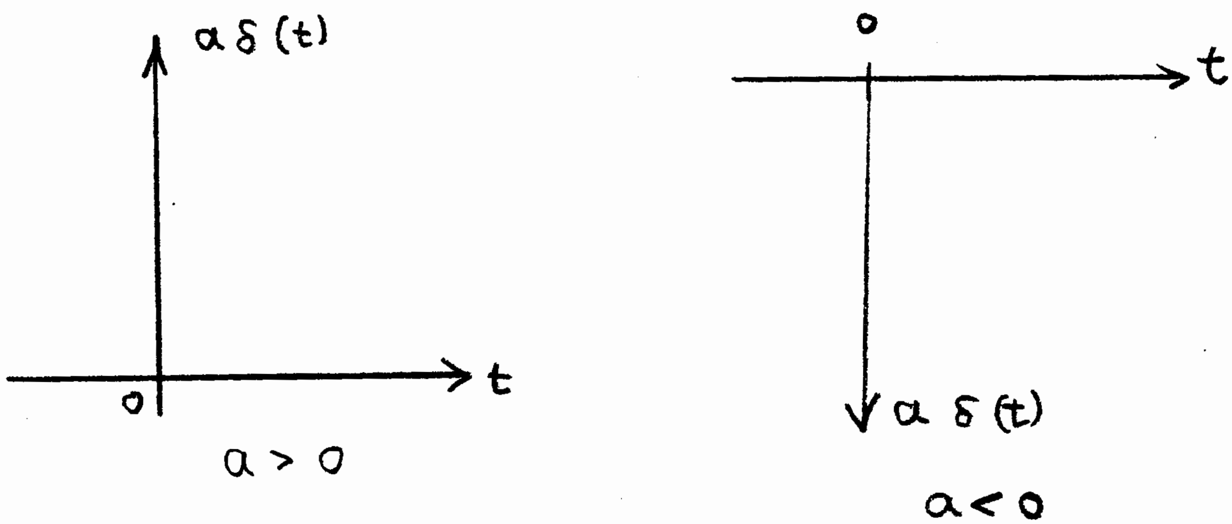
Fig. 2.2.

For a real constant a we will write $a\delta(t)$ for the function obtained from the limit

$$a\delta(t) = a \lim_{\epsilon \to 0} \delta_\epsilon(t).$$

This is likewise an impulse function but with an area a under its graph, again concentrated at the point $t = 0$. Schematically we represent this function by a "spike" which is upwards when $a > 0$ and downwards when $a < 0$. See Fig. 2.3.

Fig. 2.3.



2.1.2 The Sifting Formula

An essential property of the impulse function $\delta(t)$ is given by the so-called **sifting theorem** or **sifting formula** which reads as follows:

Theorem 2.1.1 Suppose that $x(t)$ is a continuous signal (i.e. $x(t)$ has no "jumps" or points of discontinuity). Then

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau, \qquad \text{for each instant } t. \tag{2.9}$$

To make this result plausible, regard t as fixed. Then of course $\delta(t-\tau) = 0$, for all $\tau \neq t$, thus

$$x(\tau)\delta(t-\tau) = 0 \quad \text{and} \quad x(t)\delta(t-\tau) = 0 \qquad \text{for all } \tau \neq t,$$

so that

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau), \quad \text{for all } \tau.$$

It then follows that

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau \\ &= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = x(t), \end{aligned}$$

where the last equality follows since $\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = 1$ (why?). This establishes (2.9).

Remark 2.1.2 An alternative statement of the sifting theorem is the following: for a continuous signal $x(t)$ we have

$$\int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau = x(t), \qquad \text{for each instant } t.$$

This follows by making the substitution $\sigma = t - \tau$, for then

$$\begin{aligned} \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau &= \int_{\infty}^{-\infty} x(\sigma)\delta(t-\sigma)(-d\sigma) \\ &= \int_{-\infty}^{\infty} x(\sigma)\delta(t-\sigma)d\sigma \\ &= x(t), \end{aligned}$$

where we have used Theorem 2.1.1 at the final equality.

2.1.3 High-Order Impulse Functions

Formally we can differentiate the impulse function to get the **first order impulse function** namely

$$\delta^{(1)}(t) \triangleq \frac{d\delta(t)}{dt}.$$

We can establish an analogue of the sifting theorem for the first order delta function. Indeed, let $x(t)$ be a signal which is *continuously differentiable.* This means that $x(t)$ has a derivative $x^{(1)}(t)$ at each and every instant t, and furthermore both $x(t)$ and $x^{(1)}(t)$ are continuous functions of t. Now fix some instant t and use the substitution $\sigma = t - \tau$ to get

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)\delta^{(1)}(t-\tau)d\tau &= \int_{\infty}^{-\infty} x(t-\sigma)\delta^{(1)}(\sigma)(-d\sigma) \\ &= \int_{-\infty}^{\infty} x(t-\sigma)\delta^{(1)}(\sigma)d\sigma \end{aligned} \tag{2.10}$$

Now, using integration-by-parts we have

$$\int_{-\infty}^{\infty} x(t-\sigma)\delta^{(1)}(\sigma)d\sigma = x(t-\sigma)\delta(\sigma)|_{\sigma=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dx(t-\sigma)}{d\sigma}\delta(\sigma)d\sigma. \tag{2.11}$$

Now clearly

$$\frac{dx(t-\sigma)}{d\sigma} = -x^{(1)}(t-\sigma),$$

and, since $\delta(\sigma) = 0$ when $\sigma \neq 0$ (by (2.8)), it follows

$$x(t-\sigma)\delta(\sigma)|_{\sigma=-\infty}^{\infty} = 0.$$

Putting these facts into (2.11) shows that

$$\int_{-\infty}^{\infty} x(t-\sigma)\delta^{(1)}(\sigma)d\sigma = \int_{-\infty}^{\infty} x^{(1)}(t-\sigma)\delta(\sigma) = x^{(1)}(t), \tag{2.12}$$

where Remark 2.1.2 is used at the final equality. Now use (2.11) and (2.12) to get

$$\int_{-\infty}^{\infty} x(\tau)\delta^{(1)}(t-\tau)d\tau = x^{(1)}(t),$$

which is the analogue of the sifting formula (2.9) for *first derivatives* of the signal $x(t)$. In fact, by successive application of integration-by-parts one can generalize this result to the case of n-th order derivatives of $x(t)$. To this end, introduce the n-th order derivative of $\delta(t)$, namely

$$\delta^{(n)}(t) \triangleq \frac{d^n\delta(t)}{dt^n}, \qquad n = 1, 2, 3, \ldots$$

Then a rather tedious n-times repetition of integration by parts gives the following **sifting formula for high-order derivatives:**

Theorem 2.1.3 Let $x(t)$ be a signal which is n-times continuously differentiable for some integer $n = 1, 2, \ldots$ (that is, $x(t)$ has derivatives $x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)$, at each instant t, and $x(t), x^{(1)}(t), x^{(2)}(t) \ldots, x^{(n)}(t)$ are continuous functions of t). Then we have

$$\int_{-\infty}^{\infty} x(\tau)\delta^{(n)}(t-\tau)d\tau = x^{(n)}(t), \qquad \text{for each instant } t.$$

Remark 2.1.4 From Theorem 2.1.1 we see that the impulse function, appropriately shifted by some instant t, "sifts out" the value of a continuous signal at that instant. The high order impulse functions on the other hand have a similar sifting property but now yield the *derivatives* at instant t of a sufficiently differentiable signal $x(t)$.

2.2 The Impulse Response

Definition 2.2.1 A continuous-time system (linear or not) with input $x(t)$ and output $y(t)$ is **initially at rest** when the following holds for each fixed instant t_0: if $x(t) = 0$ for all $t \leq t_0$ then $y(t) = 0$ for all $t \leq t_0$.

Intuitively, a system is initially at rest when the output, along with all of its derivatives, is *identically zero* in the "infinitely distant" past, before an input signal is applied. In accordance with this interpretation we shall use the notation

$$y(-\infty) = 0, \quad y^{(1)}(-\infty) = 0, \quad y^{(2)}(-\infty) = 0, \quad \ldots y^{(n)}(-\infty) = 0,$$

to **symbolically denote** the fact that a system is initially at rest in the sense of Definition 2.2.1.

Definition 2.2.2 The **impulse response** of a **linear time-invariant system** is the output signal when the input signal is the impulse function.

Remark 2.2.3 The impulse response is denoted by $h(t)$, after Oliver Heaviside who formulated the ideas of the impulse function and impulse response. To get a clearer understanding of the impulse response consider the following examples:

Example 2.2.4 Consider a metal ball of unit mass constrained to slide frictionlessly in a straight horizontal metal groove. This is a system for which the input signal is the force $f(t)$ applied to the ball along the direction of the groove, and the output signal is the velocity $v(t)$ of the ball along the groove. Take $f(t) > 0$ when the force is exerted from left to right along the groove and $f(t) < 0$ when the force is exerted from right to left, and likewise take $v(t) > 0$ when the ball moves from left to right and $v(t) < 0$ when the ball moves from right to left. See Fig. 2.4.

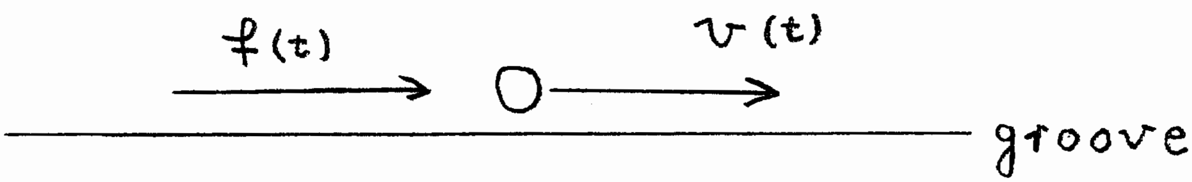


Fig. 2.4.

Suppose that the ball has stood still in the groove for all $t < 0$ and that the force $f(t)$ is applied starting at $t = 0$. From Newton's law the relation between $f(t)$ and $v(t)$ is given by

$$\begin{cases} \dot{v}(t) = f(t), \\ v(0) = 0. \end{cases} \tag{2.13}$$

The impulse response of this system is the signal $v(t)$ when the input $f(t)$ is the impulse function $\delta(t)$. Physically this represents the ball struck with a hammer which provides an incredibly sharp blow for an infinitesimal amount of time, causing the ball to move from left to right. How do we calculate the impulse response? Suppose we apply a pulse of force $f_\epsilon(t)$ having the profile shown in Fig 2.5, where $\epsilon > 0$ is a small number.

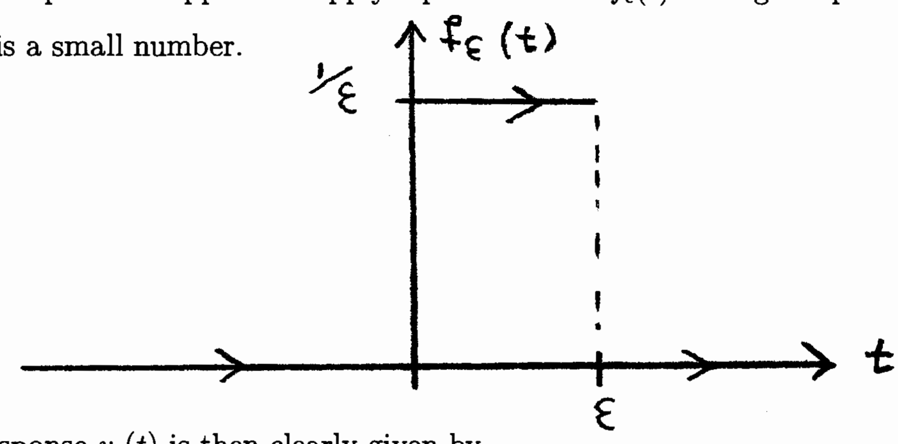


Fig. 2.5.

The corresponding response $v_\epsilon(t)$ is then clearly given by

$$v_\epsilon(t) = \begin{cases} 0, & \text{for all } t < 0, \\ t/\epsilon, & \text{for all } 0 \le t \le \epsilon, \\ 1, & \text{for all } t > \epsilon. \end{cases}$$

See Fig. 2.6.

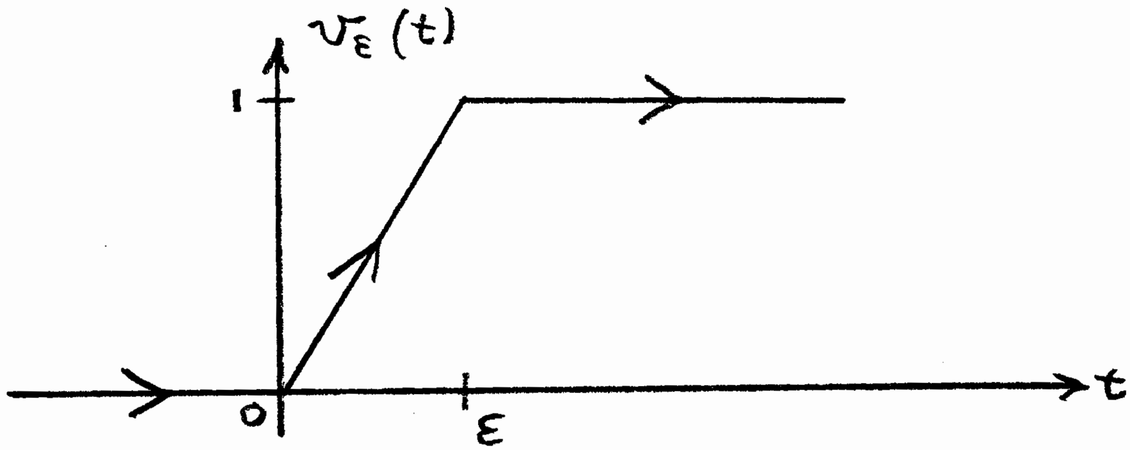


Fig. 2.6.

The impulse response $v(t)$ corresponds to the case where the parameter $\epsilon > 0$ is "infinitesimally small" which suggests that it is given by

$$v(t) = \lim_{\epsilon \to 0} v_\epsilon(t),$$

or equivalently by

$$v(t) = \begin{cases} 0, & \text{when } t < 0, \\ 1, & \text{when } t > 0. \end{cases}$$

Notice that $v(t)$ makes a *discontinuous* change at the instant $t = 0$ when the hammer falls. What is the actual value $v(0)$? The physics of the situation provides no natural anwer to this question, and we shall regard $v(0)$ as *undefined* or *indeterminate*. With this in mind, it follows that the "initial condition" $v(0) = 0$ attached to (2.13) is not quite right, since it is formulated in terms of the *undefined* quantity $v(0)$. What this initial condition should really be saying is that the ball is standing still *infinitesimally prior* to the instant $t = 0$ at which the hammer strikes, and we shall write this as $v(0-) = 0$, where $v(0-)$ is notation for the velocity infinitesimally before $t = 0$, or, in slightly more mathematical terms,

$$v(0-) \triangleq \lim_{t \to 0, t < 0} v(t).$$

It now follows that the correct formulation of the differential equation and initial condition in (2.13) is actually

$$\begin{cases} \dot{v}(t) = f(t), \\ v(0-) = 0. \end{cases}$$

Remark 2.2.5 In the preceding example we solved

$$\dot{v}(t) = \delta(t),$$

which is a differential equation with an impulse function on the right hand side. It is the presence of this impulse function that causes the jump in $v(t)$ at $t = 0$ and requires us to use the initial condition

$v(0-) = 0$ rather than the more familiar initial condition $v(0) = 0$. From now on, whenever we deal with differential equations in which impulse or high-order impulse functions occur on the right side, we must be prepared for the possibility of jumps in the solution which may occur at those instants when the impulse functions "strike".

Example 2.2.6 Consider the circuit shown in Fig. 2.7.

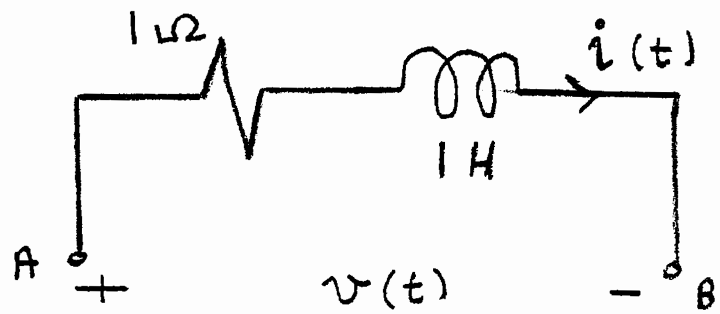


Fig. 2.7.

The input signal is the voltage $v(t)$ and the output signal is the current $i(t)$. We must calculate the impulse reponse, namely the current $i(t)$ resulting from the input signal $v(t) = \delta(t)$, when the system is initially at rest, that is $i(t) = 0$ for all $t < 0$. Physically, the impulse response describes the result of applying a voltage "spike" at the instant $t = 0$ across the terminals AB. The spike is of effectively infinite magnitude and infinitesimal duration. In order to get the impulse response we first write down the differential equation for the circuit, which is easily seen to be

$$\frac{di(t)}{dt} + i(t) = v(t), \tag{2.14}$$

and the impulse response is the solution of this equation when $v(t) = \delta(t)$. What is the correct initial condition to use for this equation? Since the circuit is initially at rest we have $i(t) = 0$ for all $t < 0$ and so it is tempting to use

$$i(0) = 0 \tag{2.15}$$

as the initial condition. However, in the light of Example 2.2.4, we know that there is the possibility of a *discontinuous jump* occurring in the solution $i(t)$ at $t = 0$, because this is the instant at which the impulse on the right hand side strikes. If this happens then the quantity $i(0)$ will be indeterminate or undefined (exactly as $v(0)$ was in the previous example) and so cannot be used to formulate the initial condition. To allow for the possibility of such a jump, we are therefore going to use

$$i(0-) = 0 \tag{2.16}$$

at the initial condition rather than (2.15), where $i(0-)$ is the current *infinitesimally prior* to instant $t=0$. This latter initial condition perfectly accounts for the physics of the situation, since $i(t)=0$ for all $t<0$, and is also consistent with the possibility of a jump occurring in $i(t)$ at $t=0$. Thus, to get the impulse response we must solve the differential equation

$$\text{(2.17)} \qquad \begin{cases} di(t)/dt + i(t) = v(t), \\ i(0-) = 0, \end{cases}$$

when $v(t) = \delta(t)$. How do we go about solving this equation? At present we do not have quite enough mathematics to directly get solutions of differential equations with impulse functions appearing on the right hand side, so let us use the indirect approach of Example 2.2.4 and consider the response of the circuit to a voltage pulse $v_\epsilon(t)$ shown in Fig. 2.8, where $\epsilon > 0$ is a small parameter.

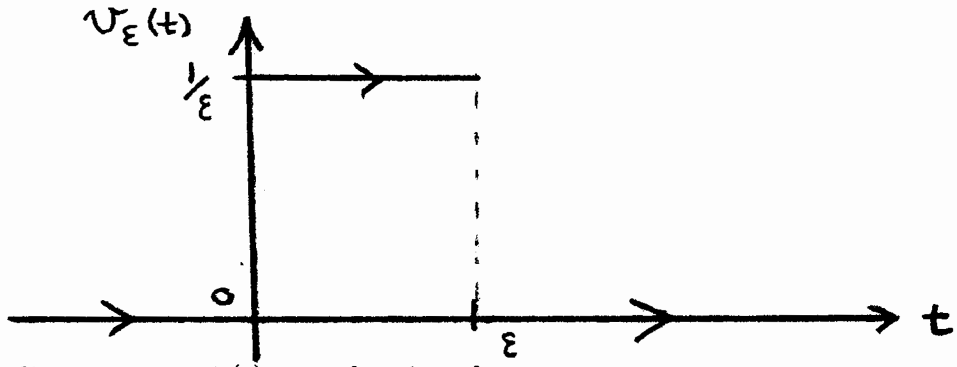


Fig. 2.8.

The corresponding response $i_\epsilon(t)$ must be given by

$$\text{(2.18)} \qquad \begin{cases} i_\epsilon(0-) = 0, & \\ di_\epsilon(t)/dt + i_\epsilon(t) = \frac{1}{\epsilon}, & \text{for all } 0 \le t \le \epsilon, \\ di_\epsilon(t)/dt + i_\epsilon(t) = 0, & \text{for all } \epsilon < t, \end{cases}$$

which are simple first order differential equations. Dealing with the first differential equation in (2.18), we easily see that

$$\text{(2.19)} \qquad i_\epsilon(t) = \frac{1}{\epsilon}(1-e^{-t}), \qquad \text{for all } 0 \le t \le \epsilon,$$

as may be checked by substitution. Over the interval $\epsilon < t$ we must solve the second differential equation in (2.18) subject to the condition

$$i_\epsilon(\epsilon) = \frac{1}{\epsilon}\left(1-e^{-\epsilon}\right),$$

and this is easily seen to be

$$\text{(2.20)} \qquad \begin{aligned} i_\epsilon(t) &= i_\epsilon(\epsilon)e^{-(t-\epsilon)}, \\ &= \frac{1}{\epsilon}(1-e^{-\epsilon})e^{-(t-\epsilon)}, \qquad \text{for all } t > \epsilon. \end{aligned}$$

See Fig. 2.9.

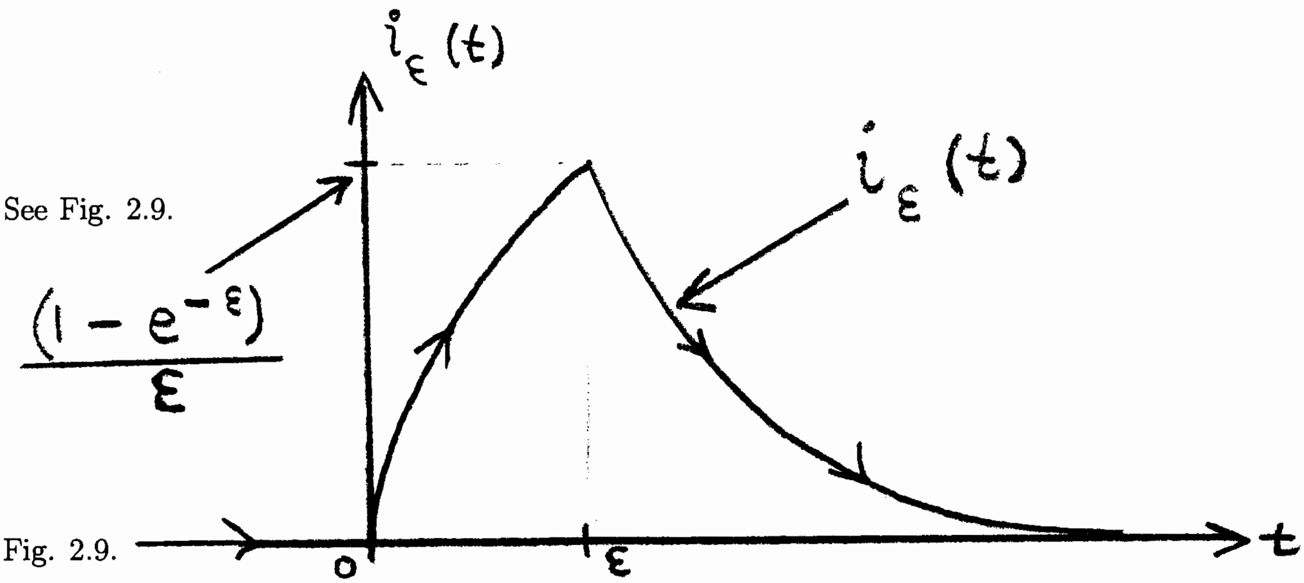


Fig. 2.9.

Now the voltage pulse $v_\epsilon(t)$ clearly tends towards an impulse function when $\epsilon \to 0$, thus we expect the impulse response to be given by

$$i(t) = \lim_{\epsilon \to 0} i_\epsilon(t). \tag{2.21}$$

By elementary properties of exponentials we know

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}(1 - e^{-\epsilon}) = 1,$$

thus it follows from (2.20) and (2.21) that

$$\begin{aligned} i(t) &= \lim_{\epsilon \to 0} i_\epsilon(t) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}(1 - e^{-\epsilon})e^{-(t-\epsilon)} \\ &= e^{-t}, \qquad \text{for all } t > 0. \end{aligned} \tag{2.22}$$

Combining this fact with the fact that $i(t) = 0$ for all $t < 0$ gives the complete impulse response as follows:

$$i(t) = \begin{cases} 0, & \text{for all } t < 0, \\ e^{-t}, & \text{for all } t > 0. \end{cases}$$

We see that there is indeed a jump in $i(t)$ at $t = 0$, as expected, and thus the value $i(0)$ is *indeterminate.*

Remark 2.2.7 The preceding two examples illustrate a technique for determining the impulse response, namely approximate the impulse by a tall narrow pulse of unit area, evaluate the system response for this pulse, and then take limits as the pulse becomes infinitely "thin" and "high". It is clear that this approach will become rather cumbersome, even for systems given by only moderately complicated differential equations. In the next chapter we shall see that *Laplace Transforms* provide an extremely powerful method for getting the impulse response for a very general class of systems. Putting aside the question of how to compute the impulse response, we next give a result which shows the usefulness of the impulse response when we are dealing with *linear time invariant systems:*

Theorem 2.2.8 Suppose that $h(t)$ is the impulse response of an LTI system. Then, for an input signal $x(t)$, the response of the system is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \tag{2.23}$$

for each instant t, provided that the integral in (2.23) exists.

Remark 2.2.9 This result shows that if we know the impulse response $h(t)$ of an LTI system then we can determine the response of the system to *any* input signal $x(t)$.

A proof of this theorem is definitely beyond the scope of this course but it can be made plausible by the following argument: Let $x(t)$ be a given input signal. For the function $\delta_\epsilon(t)$ given by (2.1) we see from Fig. 2.1 that the function $\delta_\epsilon(t-t_0)$, which is $\delta_\epsilon(t)$ shifted by a fixed instant t_0, is as shown in Fig. 2.10:

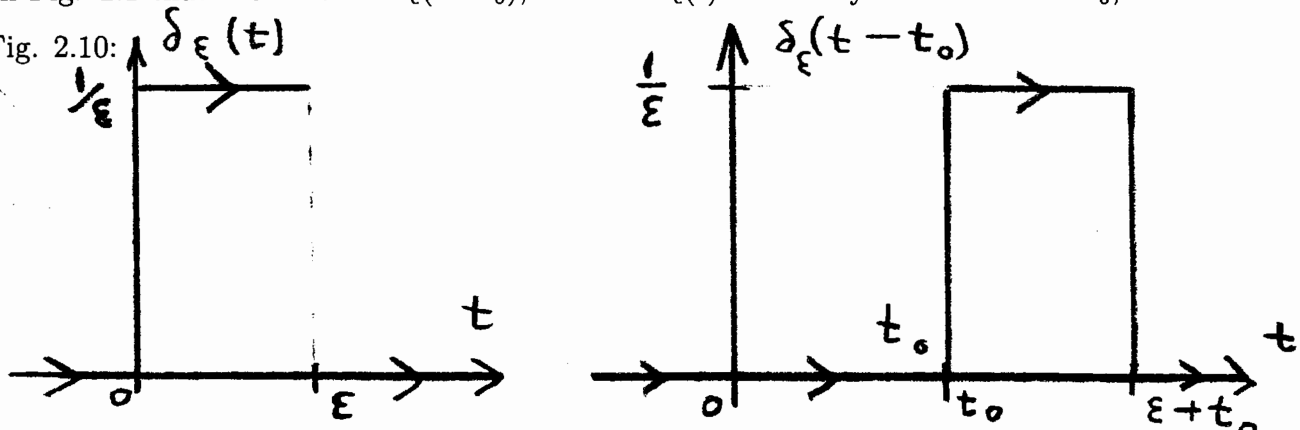


Fig. 2.10.

Now consider a given signal $x(t)$, and let $x_\epsilon(t)$ be the "staircase" approximation to this signal shown in Fig. 2.11:

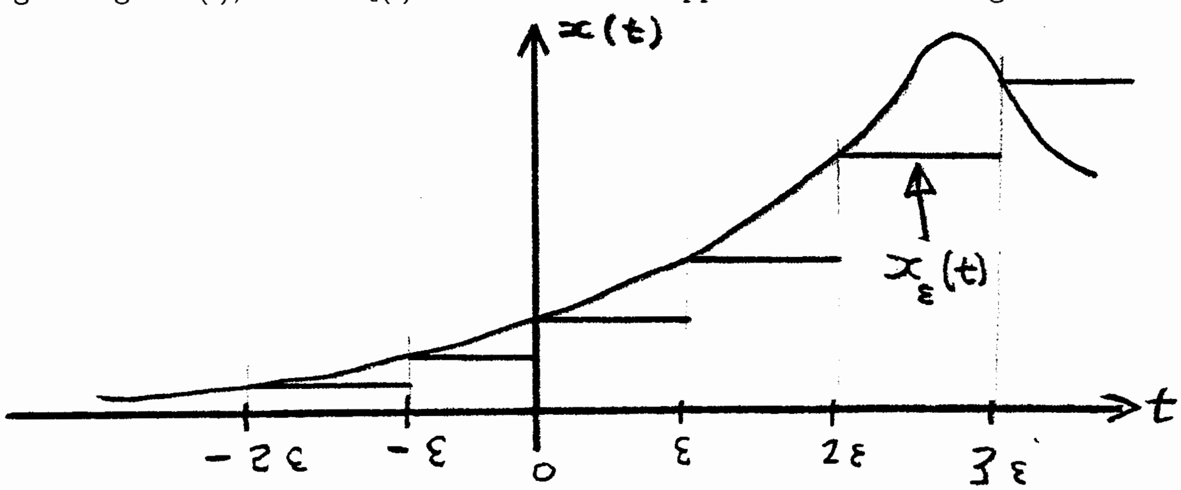


Fig. 2.11.

It is clear from the figure that for each instant t we have

(2.24)
$$x_\epsilon(t) = \sum_{k=-\infty}^{\infty} x(k\epsilon)\delta_\epsilon(t - k\epsilon)\epsilon.$$

Moreover, as $\epsilon \to 0$, one sees from Fig. 2.11 that $x_\epsilon(t)$ tends towards $x(t)$, namely

(2.25)
$$x(t) = \lim_{\epsilon \to 0} x_\epsilon(t).$$

Let $h_\epsilon(t)$ be the response of the LTI system to the input $\delta_\epsilon(t)$. Then, by time invariance of the system, we see that the response to the shifted input signal $\delta_\epsilon(t - k\epsilon)$ is the correspondingly shifted output signal $h_\epsilon(t - k\epsilon)$, so that, by linearity, the response to the scaled input signal $x(k\epsilon)\delta_\epsilon(t - k\epsilon)$ must be the correspondingly scaled output signal $x(k\epsilon)h_\epsilon(t - k\epsilon)$. Thus, again by linearity, the response of the system to the input signal $x_\epsilon(t)$ given by the sum of signals in (2.24), must be the signal $y_\epsilon(t)$ given by the sum of the corresponding output signals, that is

(2.26)
$$y_\epsilon(t) \triangleq \sum_{k=-\infty}^{\infty} x(k\epsilon)h_\epsilon(t - k\epsilon)\epsilon.$$

Now let $y(t)$ be the response of the system to the input signal $x(t)$. Since the input signals $x_\epsilon(t)$ tend towards the signal $x(t)$ as $\epsilon \to 0$ (see (2.25)), it is plausible that the corresponding output signals $y_\epsilon(t)$ should converge to $y(t)$, namely

(2.27)
$$y(t) = \lim_{\epsilon \to 0} y_\epsilon(t).$$

By (2.27) and (2.26),

$$\begin{aligned} y(t) &= \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} x(k\epsilon)h_\epsilon(t - k\epsilon)\epsilon \\ &= \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau, \end{aligned}$$

which is (2.23).

2.2.1 Causal LTI Systems

The expression (2.23) holds for an LTI system which is not necessarily causal. If the system is also causal, as well as LTI, then the impulse response has an additional property that we now establish. We know from linearity and Remark 1.2.7 that the response $y_1(t)$ to the *zero* input signal $x_1(t)$ is itself the zero signal, namely

$$y_1(t) = 0, \qquad \text{for all instants } t. \tag{2.28}$$

Next, consider the input signal $x_2(t)$ which is the impulse function $\delta(t)$, in which case the corresponding output signal $y_2(t)$ is the system impulse response $h(t)$. Fix some arbitrary $t_0 < 0$. Then, since $x_1(t)$ is the zero function and $x_2(t) = \delta(t) = 0$ for all $t < 0$ (see (2.8)), we certainly have

$$x_1(t) = x_2(t), \qquad \text{for all } t \leq t_0.$$

In view of causality this implies that $y_1(t_0) = y_2(t_0)$. But $y_2(t_0) = h(t_0)$ and (2.28) shows $y_1(t_0) = 0$. It follows that $h(t_0) = 0$. Thus, by the arbitrary choice of $t_0 < 0$ we see that $h(t) = 0$ for all $t < 0$. We have therefore established

$$\text{if an LTI system is causal then} \qquad h(t) = 0, \qquad \text{for all } t < 0.$$

In this case we have $h(t-\tau) = 0$ when $t - \tau < 0$, that is when $\tau > t$, and it follows that (2.23) takes the following form for an LTI causal system:

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau, \qquad \text{for all } t. \tag{2.29}$$

Moreover, it is clear that if the impulse reponse $h(t)$ of an LTI system has the property that $h(t) = 0$ for all $t < 0$, then $h(t-\tau) = 0$ when $\tau > t$, thus (2.23) reduces to (2.29), from which it follows at once that the system is causal. We therefore conclude the following for an LTI system with impulse response $h(t)$:

$$\text{the system is causal if and only if } h(t) = \text{ for all } t < 0,$$

in which case the input $x(t)$ and output $y(t)$ are related by (2.29).

2.2.2 Convolution Identities

The integral which occurs in (2.24) is called a **convolution integral**. Here we look at the problem of evaluating integrals of this kind. For two signals $x(t)$ and $y(t)$ we use the notation

$$(x * y)(t) \triangleq \int_{-\infty}^{+\infty} x(\tau)y(t-\tau)d\tau, \tag{2.30}$$

to denote the **convolution** of $x(t)$ with $y(t)$. Using this notation we can rewrite (2.24) in abbreviated form as follows:

$$y(t) = (x * h)(t).$$

The evaluation of convolution integrals is often facilitated by the following elementary facts whuch can be established by easy although tedious calculus: For signals $x(t)$, $y(t)$ and $z(t)$ we have

commutativity: $(x * y)(t) = (y * x)(t)$.

distributivity: $(x * (y + z))(t) = (x * y)(t) + (x * z)(t)$.

associativity: $(x * (y * z))(t) = ((x * y) * z)(t)$.

We next consider how to evaluate the convolution of one signal $x(t)$ with another signal $y(t)$. In principle this involves evaluating the integral on the right side of (2.30) with respect to variable τ separately for each and every instant t. In practice the problem is usually simpler than this if we can take advantage of any special structure that may be present in the signals. The key to this approach is the method of **indicator functions** which we next consider:

For constants $\alpha \leq \beta$, let $I[\alpha, \beta](\tau)$ be the function of τ defined by

$$I[\alpha, \beta](\tau) \triangleq \begin{cases} 1 & \text{when } \alpha \leq \tau \leq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

We call this the **indicator function** of the interval $\alpha \leq \tau \leq \beta$, because it takes unit value when τ belongs to this interval but is zero everywhere else. See Fig. 2.13.

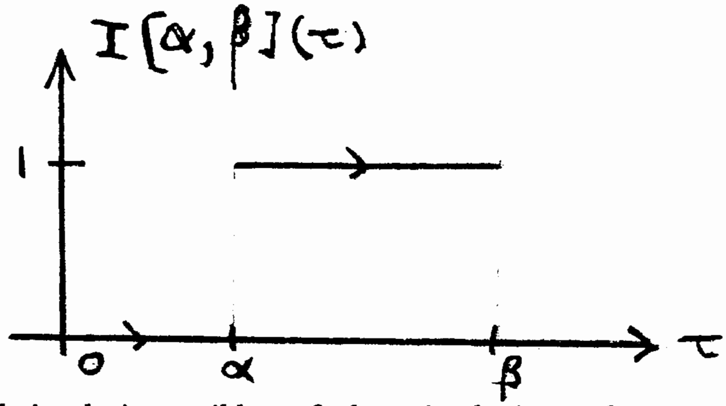


Fig. 2.13

When $\alpha > \beta$ then it is obviously impossible to find a τ in the interval $\alpha \leq \tau \leq \beta$ and in this case we put

$$I[\alpha, \beta](\tau) \triangleq 0, \qquad \text{for all } \tau.$$

The following observations about indicator functions are completely trivial:

(I.1) For constants α_1, β_1, α_2 and β_2 we have

$$I[\alpha_1, \beta_1](\tau) I[\alpha_2, \beta_2](\tau) = I[\max(\alpha_1, \alpha_2), \min(\beta_1, \beta_2)](\tau).$$

(I.2) For a signal $x(t)$ and constants $\alpha \leq \beta$ we have:

$$\int_{-\infty}^{+\infty} x(\tau) I[\alpha, \beta](\tau) d\tau = \int_{\alpha}^{\beta} x(\tau) d\tau.$$

(I.3) For a signal $x(t)$ and constants $\alpha > \beta$ we have:

$$\int_{-\infty}^{+\infty} x(\tau) I[\alpha, \beta](\tau) d\tau = 0.$$

Finally, to implement the method of indicator functions, we need a trick which is best illustrated by a simple example: Suppose a signal $y(t)$ has the form

$$y(\tau) \triangleq \begin{cases} \cos(\omega\tau) & \text{when } -5 \leq \tau \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

Fix some instant t and consider the function of τ given by $y(t - \tau)$, which we are going to write in terms of an indicator function of τ. We clearly have

$$y(t - \tau) \triangleq \begin{cases} \cos(\omega(t - \tau)) & \text{when } -5 \leq t - \tau \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

But

$$-5 \leq t - \tau \leq 10 \qquad \text{if and only if} \qquad t - 10 \leq \tau \leq t + 5,$$

so that we have

$$y(t-\tau) \triangleq \begin{cases} \cos(\omega(t-\tau)) & \text{when } t-10 \le \tau \le t+5, \\ 0 & \text{otherwise,} \end{cases}$$

and therefore

$$y(t-\tau) = \cos(\omega(t-\tau))I[t-10, t+5](\tau).$$

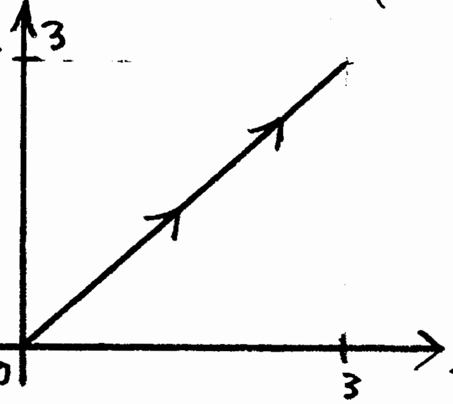
With these facts in mind we can evaluate the convolution integrals of simple signals:

Example 2.2.10 Find $(x * y)(t)$ for the signals $x(t)$ and $y(t)$ given by

$$x(t) \triangleq \begin{cases} t & \text{when } 0 \le t \le 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$y(t) \triangleq \begin{cases} 1 & \text{when } -1 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

See Fig. 2.14.



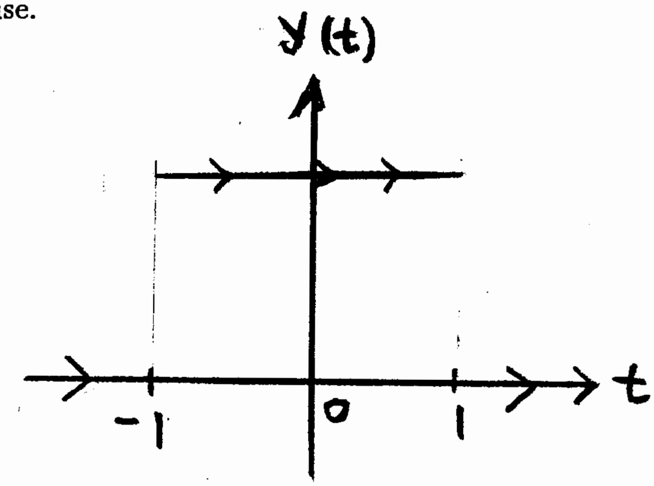


Fig. 2.14.

Now

(2.31) $$(x * y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau.$$

Clearly we have

(2.32) $$x(\tau) = \tau I[0,3](\tau).$$

Next, consider $y(t-\tau)$ as a function of τ for a fixed instant t. Clearly

$$y(t-\tau) \triangleq \begin{cases} 1 & \text{when } -1 \le t-\tau \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

and since

$$-1 \le t-\tau \le 1 \quad \text{if and only if} \quad t-1 \le \tau \le t+1,$$

we have

$$y(t-\tau) \triangleq \begin{cases} 1 & \text{when } t-1 \leq \tau \leq t+1, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently

(2.33) $$y(t-\tau) = I[t-1, t+1](\tau).$$

Combining (2.32) and (2.33), and using (I.1) gives

(2.34) $$\begin{aligned} x(\tau)y(t-\tau) &= \tau I[0,3](\tau)I[t-1,t+1](\tau) \\ &= \tau I[\max(0,t-1), \min(3,t+1)](\tau). \end{aligned}$$

For each fixed t, put

$$\alpha(t) \triangleq \max(0, t-1), \qquad \beta(t) \triangleq \min(3, t+1).$$

Then (2.34) becomes

$$x(\tau)y(t-\tau) = \tau I[\alpha(t), \beta(t)](\tau),$$

so that (2.31) in turn becomes

(2.35) $$(x * y)(t) = \int_{-\infty}^{\infty} \tau I[\alpha(t), \beta(t)](\tau) d\tau.$$

Now sketch $\alpha(t)$ and $\beta(t)$ as in Fig. 2.15.

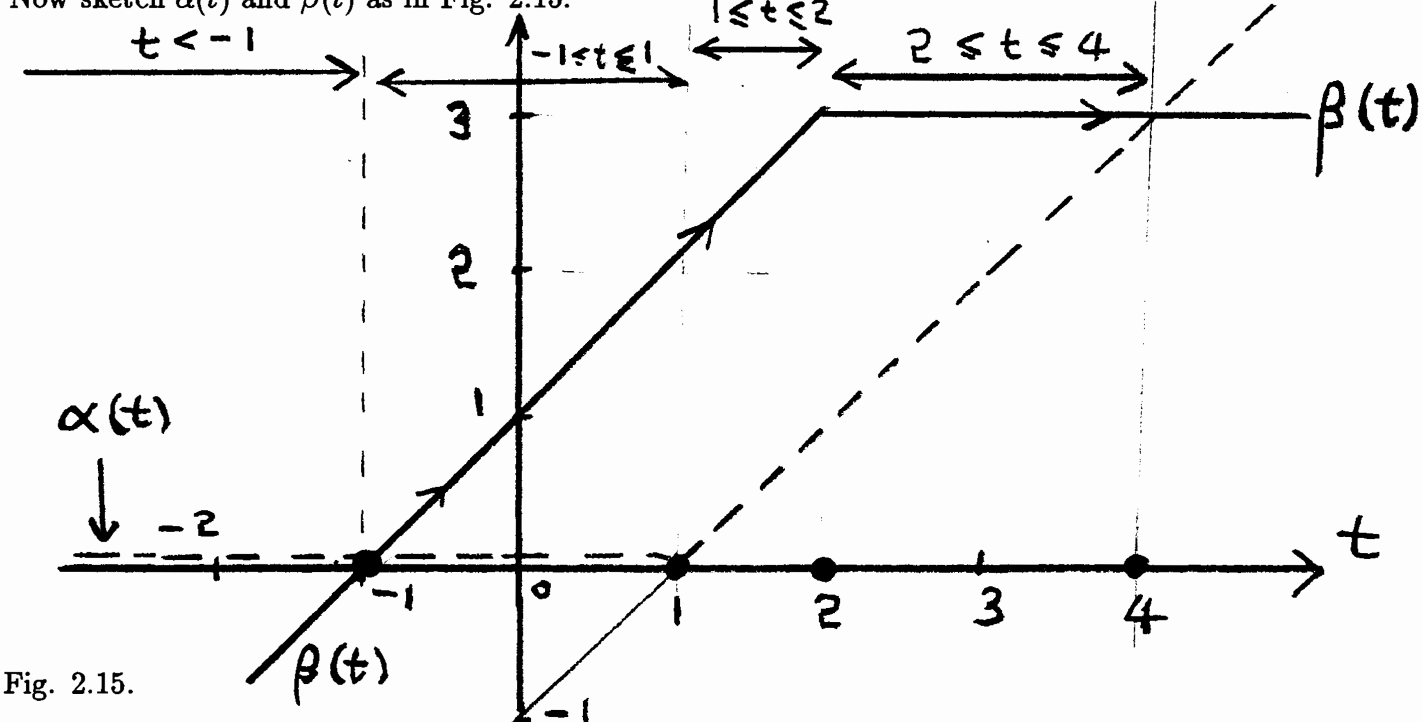


Fig. 2.15.

As shown in Fig. 2.15, things break up naturally into five sectors namely

Sector 1: Here $t < -1$, thus $\alpha(t) > \beta(t)$, hence from (I.3) and (2.35) we get

$$(x * y)(t) = 0.$$

Sector 2: Here $-1 \leq t \leq 1$, thus $\alpha(t) \leq \beta(t)$ with

$$\alpha(t) = 0, \qquad \beta(t) = t + 1,$$

so, from (I.2), we get

$$(x * y)(t) = \int_{\alpha(t)}^{\beta(t)} \tau d\tau = \int_{0}^{t+1} \tau \, d\tau = \frac{(t+1)^2}{2}.$$

Sector 3: Here $1 \leq t \leq 2$. Thus $\alpha(t) \leq \beta(t)$ with

$$\alpha(t) = t - 1, \qquad \beta(t) = t + 1,$$

so, from (I.2), we get

$$(x * y)(t) = \int_{\alpha(t)}^{\beta(t)} \tau d\tau = \int_{t-1}^{t+1} \tau \, d\tau = 2t.$$

Sector 4: Here $2 \leq t \leq 4$. Thus $\alpha(t) \leq \beta(t)$ with

$$\alpha(t) = t - 1, \qquad \beta(t) = 3,$$

so, from (I.2), we get

$$(x * y)(t) = \int_{\alpha(t)}^{\beta(t)} \tau d\tau = \int_{t-1}^{3} \tau \, d\tau = \frac{-t^2}{2} + t + 4.$$

Sector 5: Here $t > 4$. Thus $\alpha(t) > \beta(t)$ so that

$$(x * y)(t) = 0.$$

To summarize:

$$(x * y)(t) \triangleq \begin{cases} 0 & \text{when } t < -1, \\ \frac{(t+1)^2}{2} & \text{when } -1 \leq t \leq 1, \\ 2t & \text{when } 1 \leq t \leq 2, \\ \frac{-t^2}{2} + t + 4 & \text{when } 2 \leq t \leq 4, \\ 0 & \text{when } t > 4. \end{cases}$$

2.3 Linear Differential Equations

Some of the most important examples of linear time invariant systems arise from the linear differential equation with constant coefficients of the form

(2.36)
$$\frac{d^n y(t)}{d^n t} + a_{n-1}\frac{d^{n-1}y(t)}{d^{n-1}t} \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m x(t)}{d^m t} + b_{m-1}\frac{d^{m-1}x(t)}{d^{m-1}t} \ldots + b_1\frac{dx(t)}{dt} + b_0 x(t),$$

where n is a positive integer called the **order** of the equation, m is a nonnegative integer, and $b_m \neq 0$. In this equation we shall regard $x(t)$ as the input signal and $y(t)$ as the output signal. One can think of (2.36) as giving the response $y(t)$ *implicitly* in terms of its derivatives and their relation to the derivatives of the input signal $x(t)$. The reason for the importance of (2.36) is that many physical systems are characterized by equations of just this form. We have already seen such equations in Examples 1.2.21 and 1.2.22. Here are two further examples:

Example 2.3.1 Consider the circuit in Fig. 2.16:

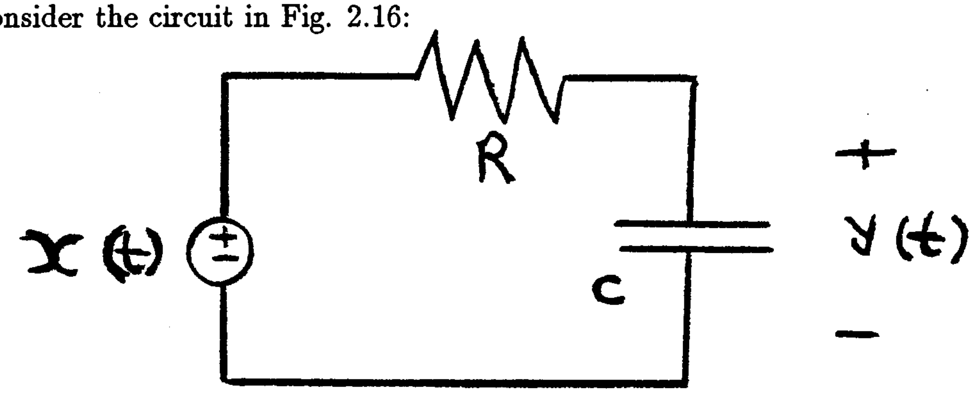


Fig. 2.16.

The input is the applied voltage $x(t)$ and the output is the voltage $y(t)$ accross the capacitor. By KVL we get

(2.37)
$$x(t) = i(t)R + y(t).$$

By the capacitor relation,

$$i(t) = C\frac{dy(t)}{dt},$$

and using this in (2.37) gives

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = x(t),$$

which is of the form (2.36) with $n = 1$, $m = 0$.

Example 2.3.2 Consider the circuit in Fig. 2.17.

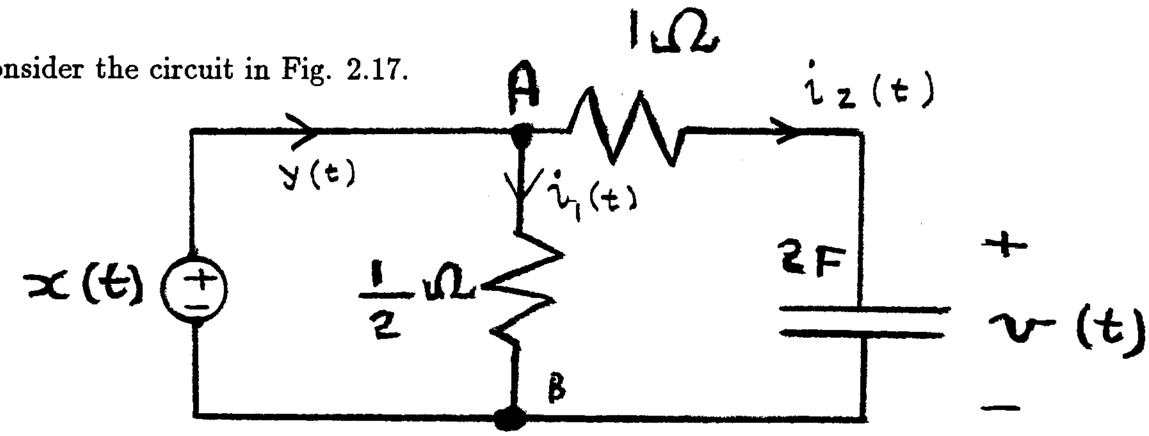


Fig. 2.17.

The input is the applied voltage $x(t)$ and the output is the current $y(t)$ through the voltage source.

By KVL:

(2.38) $$x(t) = i_2(t) + v(t).$$

By the capacitor relation we have

$$v(t) = v(0) + \frac{1}{2}\int_0^t i_2(\tau)d\tau,$$

thus (2.38) becomes

$$x(t) = i_2(t) + v(0) + \frac{1}{2}\int_0^t i_2(\tau)d\tau,$$

and taking derivatives of this equation then gives

(2.39) $$\frac{dx(t)}{dt} = \frac{di_2(t)}{dt} + \frac{i_2(t)}{2}.$$

By KVL we have

$$x(t) = \frac{1}{2}i_1(t)$$

and from KCL at node A we get

$$y(t) = i_1(t) + i_2(t)$$

so that

(2.40) $$i_2(t) = y(t) - 2x(t).$$

using (2.40) to eliminate $i_2(t)$ from (2.39) gives

$$\frac{dx(t)}{dt} = \frac{dy(t)}{dt} - 2\frac{dx(t)}{dt} + \frac{y(t)}{2} - x(t)$$

hence

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = 3\frac{dx(t)}{dt} + x(t).$$

This is of the form (2.36) with $n = m = 1$.

To study (2.36) we are going to use the following so-called D-operator notation: put

$$D^r y(t) \triangleq \frac{d^r y(t)}{dt}, \qquad D^r x(t) \triangleq \frac{d^r x(t)}{dt}, \tag{2.41}$$

for nonnegative integers r, and let

$$Q(D) \triangleq D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0, \tag{2.42}$$

$$P(D) \triangleq b_m D^m + b_{m-1}D^{m-1} + \ldots b_1 D + b_0. \tag{2.43}$$

One can think of $Q(D)$ and $P(D)$ as polynomials whose argument is the D-operator. With this notation we have

$$Q(D)y(t) = \frac{d^n y(t)}{d^n t} + a_{n-1}\frac{d^{n-1}y(t)}{d^{n-1}t} \ldots + a_1 \frac{dy(t)}{dt} + a_0,$$

and similarly for $P(D)x(t)$. Thus (2.36) can be written in compact form as

$$Q(D)y(t) = P(D)x(t). \tag{2.44}$$

Remark 2.3.3 We shall always suppose for (2.36) and (2.44) that $m \leq n$, since the case where $m > n$ turns out to be of no physical interest.

2.3.1 Systems and Linear Differential Equations

In order to regard (2.44) as a *system* with input signal $x(t)$ and output signal $y(t)$ we must make sure that the input gives rise to a *uniquely defined* output. However, (2.44) by itself is not enough to do this. In fact, to make sure that the $y(t)$ which solves (2.44) is uniquely specified we must add *auxiliary conditions* of the form

$$y(t_0) = \alpha_0, \quad y^{(1)}(t_0) = \alpha_1, \ldots, \quad y^{(n-1)}(t_0) = \alpha_{n-1},$$

where t_0 is a fixed instant and the α_i are constants. In view of our experience with Examples 2.2.4 and 2.2.6 the question immediately arises: are these auxiliary conditions completely appropriate? To see the problem which could arise consider Example 2.3.2 in which we have the equation

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = 3\frac{dx(t)}{dt} + x(t).$$

Suppose the input signal is given by

$$x(t) = u(t-1), \tag{2.45}$$

(i.e. a delayed unit step) and we want to apply the auxiliary condition

$$y(1) = 2. \tag{2.46}$$

For the signal in (2.45) we have

$$\begin{aligned} 3\frac{dx(t)}{dt} + x(t) &= 3\frac{du(t-1)}{dt} + u(t-1) \\ &= 3\delta(t-1) + u(t-1), \end{aligned} \tag{2.47}$$

so we are really solving the equation

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = 3\delta(t-1) + u(t-1)$$

subject to the condition (2.46). The difficulty is that the term $\delta(t-1)$ on the right hand side will cause a discontinuity in $y(t)$ at instant $t = 1$, exactly as happened in Examples 2.2.4 and 2.2.6, so that $y(1)$ will be indeterminate. For this reason we must replace (2.46) with a condition which holds *infinitesimally prior* to the instant $t = 1$, namely

$$y(1-) = 2.$$

Abstracting this argument to the general case of (2.44) we shall henceforth use auxiliary conditions of the form

$$y(t_0-) = \alpha_0, \quad y^{(1)}(t_0-) = \alpha_1, \ldots, \quad y^{(n-1)}(t_0-) = \alpha_{n-1},$$

which hold infinitesimally prior to the instant $t = t_0$. From the theory of ordinary differential equations it is known that, for a given input signal $x(t)$, there is *exactly one* signal $y(t)$ which both satisfies (2.44) and these auxiliary conditions. This means that we can regard

$$\text{(2.48)} \qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(t_0-) = \alpha_0, \quad y^{(1)}(t_0-) = \alpha_1, \ldots, \quad y^{(n-1)}(t_0-) = \alpha_{n-1}, \end{cases}$$

as a *system* with input signal $x(t)$ and output signal $y(t)$. As the constants α_i and the instant t_0 are changed we get *different* systems from (2.48).

The main system property of interest in this course is linearity, so let us see whether this system is linear. Suppose that some of the constants α_i are *non-zero*, and let the input $x(t)$ be the zero signal. From the auxiliary conditions in (2.48) we see that $y^{(i)}(t_0-) = \alpha_i \neq 0$ for some i, which means that the output $y(t)$ corresponding to the input zero signal *cannot* be the zero signal. Since the output of a linear system is always the zero signal when the input is the zero signal (see Remark 1.2.7) it follows that system (2.48) is **not** linear when $\alpha_i \neq 0$ for some i. Next, suppose that all of the α_i are zero, so that we are dealing with the system

$$\text{(2.49)} \qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(t_0-) = 0, \quad y^{(1)}(t_0-) = 0, \ldots, \quad y^{(n-1)}(t_0-) = 0, \end{cases}$$

where t_0 is some fixed instant. Let $x_1(t)$ and $x_2(t)$ be input signals for (2.49) with corresponding output signals $y_1(t)$ and $y_2(t)$. Thus

$$\begin{cases} Q(D)y_i(t) = P(D)x_i(t), \\ y_i(t_0-) = 0, \quad y_i^{(1)}(t_0-) = 0, \ldots, \quad y_i^{(n-1)}(t_0-) = 0, \end{cases}$$

for $i = 1, 2$, so that linearity of derivatives gives

$$\begin{cases} Q(D)(y_1 + y_2)(t) = P(D)(x_1 + x_2)(t), \\ y_1(t_0-) + y_2(t_0-) = 0, \quad y_1^{(1)}(t_0-) + y_2^{(1)}(t_0-) = 0, \ldots, \quad y_1^{(n-1)}(t_0-) + y_2^{(n-1)}(t_0-) = 0, \end{cases}$$

showing that $y_1(t) + y_2(t)$ is the output corresponding to the input $x_1(t) + x_2(t)$. In the same way we see that, for any constant c, the input $cx_1(t)$ causes the output $cy_1(t)$, showing that (2.49) is indeed linear. We conclude that (2.48) is a linear system if and only if all of the α_i are zero.

Having established linearity of (2.49) let us now determine if it is causal. To get some understanding of this question consider the following simple example

$$
\left\{
\begin{array}{l}
\dot{y}(t) + 2y(t) = x(t), \\
y(0-) = 0,
\end{array}
\right. \tag{2.50}
$$

which is clearly a special case of (2.49). Suppose the input signal is

$$
x_1(t) \triangleq \left\{
\begin{array}{ll}
0 & \text{when } t \leq -1, \\
1 & \text{otherwise.}
\end{array}
\right.
$$

and let $y_1(t)$ be the corresponding output, namely

$$
\left\{
\begin{array}{l}
\dot{y}_1(t) + 2y_1(t) = x_1(t), \\
y_1(0-) = 0.
\end{array}
\right.
$$

The solution of this equation is given by

$$
y_1(t) = \left\{
\begin{array}{ll}
\frac{1}{2}\left(e^{-2(t+1)} - e^{-2t}\right), & \text{when } t < -1, \\
\frac{1}{2}\left(1 - e^{-2t}\right), & \text{when } t \geq -1,
\end{array}
\right.
$$

as may easily be seen by direct substition. The graph of $y_1(t)$ is shown in Fig. 2.18.

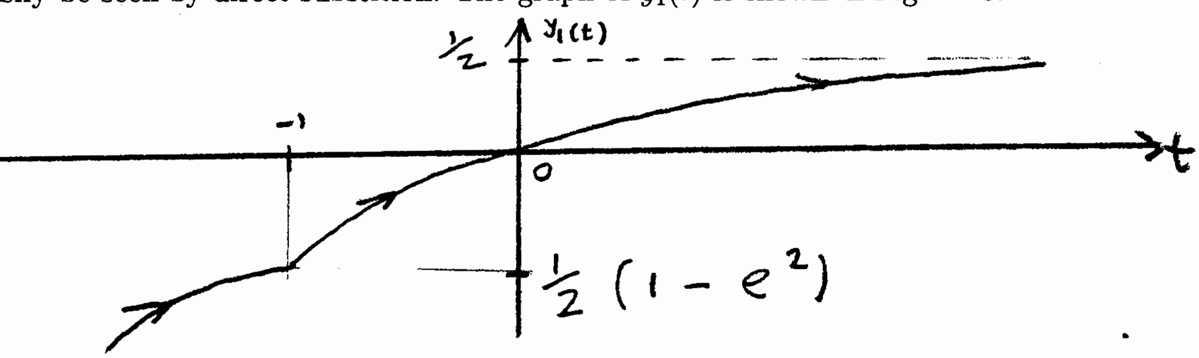


Fig. 2.18.

Next, let $x_2(t)$ be the zero signal, and let $y_2(t)$ be the corresponding output of (2.50). Since this has been seen to be a linear system it follows from Remark 1.2.7 that $y_2(t)$ must also be the zero signal. Now we clearly have

$$
x_1(t) = x_2(t), \qquad \text{for all } t \leq -1,
$$

so that if (2.50) is causal then we must have

$$
y_1(t) = y_2(t), \qquad \text{for all } t \leq -1,
$$

as follows from the very definition of causality (see Definition 1.2.16). This in turn implies that $y_1(t) = 0$ for all $t \leq -1$, which is clearly not the case, as one sees from Fig. 2.18. It follows that the linear system (2.50) *cannot* be causal.

Remark 2.3.3 Since causality fails for such a simple special instance of (2.49) it follows that causality cannot hold in general for the linear system (2.49). The problem with this example is that the auxiliary condition $y(0) = 0$ effectively forces the output of the system to "anticipate" the future for *negative values* of t in such a way that the auxiliary condition $y(0) = 0$ is satisfied. This suggests that if the system is initially at rest (recall Definition 2.2.1), the intuitive significance of which is that "zero auxiliary conditions" are effectively applied at $t_0 = -\infty$, then perhaps causality of the system will indeed result. As noted after Definition 2.2.1, we symbolically represent the fact that a system is initially at rest by the notation,

$$y(-\infty) = 0, \quad y^{(1)}(-\infty) = 0, \ldots, \quad y^{(n-1)}(-\infty) = 0,$$

an therefore we will use

$$\text{(2.51)} \qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(-\infty) = 0, \quad y^{(1)}(-\infty) = 0, \ldots, \quad y^{(n-1)}(-\infty) = 0, \end{cases}$$

to denote the system (2.44) when it is initially at rest.

We are now going to show that (2.51) is linear, causal and time invariant, but prior to establishing this we make the following simple observation: If

(i) $x(t)$ is an input signal such that $x(t) = 0$ for all $t \leq t_0$,

(ii) the system (2.44) is initially at rest,

then the response $y(t)$ is such that $y(t) = 0$ for all $t \leq t_0$ (see Fig. 2.19). As for $y(t)$ when $t > t_0$, this is just given by the solution of the linear differential equation

$$Q(D)y(t) = P(D)x(t), \qquad t \geq t_0,$$

subject to the auxiliary conditions

$$y(t_0) = 0, \quad y^{(1)}(t_0) = 0, \ldots, \quad y^{(n-1)}(t_0) = 0.$$

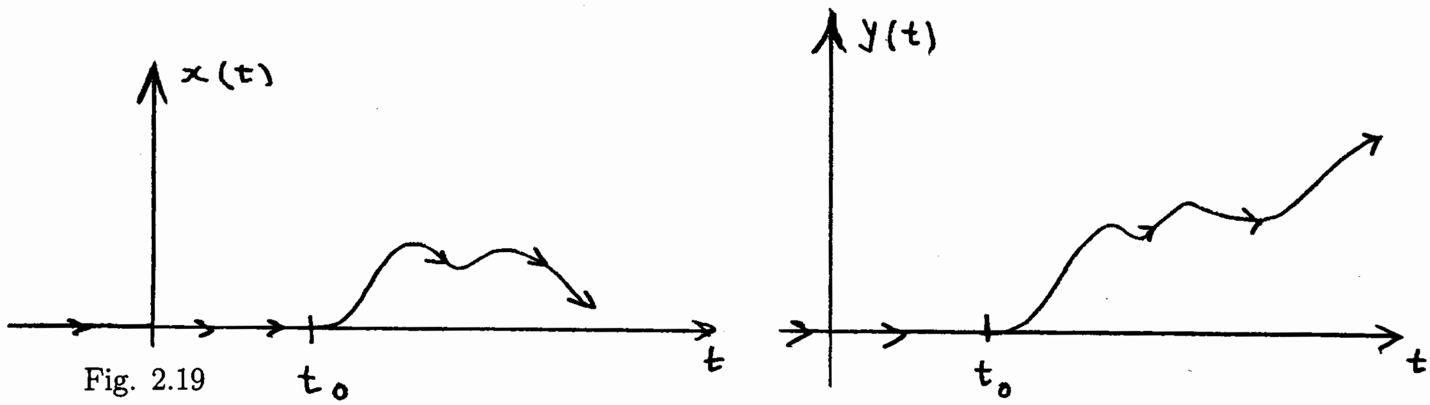


Fig. 2.19

We now establish the following:

Theorem 2.3.4 The initially-at-rest system (2.51) is linear, causal, and time invariant.

Proof: Linearity follows in exactly the same way that we showed linearity of (2.49). To see causality let $x_1(t)$ and $x_2(t)$ be inputs with corresponding outputs $y_1(t)$ and $y_2(t)$ respectively, such that

$$x_1(t) = x_2(t), \qquad \text{for all } t \leq t_0 \tag{2.52}$$

for some instant t_0. We must show that

$$y_1(t_0) = y_2(t_0), \tag{2.53}$$

to establish that the system is causal. Put

$$x(t) \triangleq x_1(t) - x_2(t), \qquad y(t) \triangleq y_1(t) - y_2(t). \tag{2.54}$$

It follows by linearity of (2.51) that $y(t)$ is the response to the input signal $x(t)$, and (2.52) implies that

$$x(t) = 0 \qquad \text{for all } t \leq t_0.$$

Since (2.51) is initially at rest (Definition 2.2.1), we then see that

$$y(t) = 0 \qquad \text{for all } t \leq t_0,$$

so that (2.53) follows, as required for causality.

It remains to show that (2.51) is time invariant. With the mathematical tools available to us we cannot give a completely water-tight demonstration of this fact, but we can make time invariance very plausible by arguing as follows: Let $x_1(t)$ be an input signal with corresponding output $y_1(t)$, and fix some t_0. Then

$$\text{(2.55)} \qquad \begin{cases} Q(D)y_1(t) = P(D)x_1(t), \qquad \text{for all } t, \\ y_1(-\infty) = 0, \quad y_1^{(1)}(-\infty) = 0, \ldots, \quad y_1^{(n-1)}(-\infty) = 0. \end{cases}$$

Put

$$\text{(2.56)} \qquad x_2(t) \triangleq x_1(t - t_0), \qquad\qquad y_2(t) \triangleq y_1(t - t_0).$$

It remains to show that $y_2(t)$ is the output that corresponds to the input signal $x_2(t)$ in order to see that time invariance follows. Since the differential equation of (2.55) holds for all t in the range $-\infty < t < \infty$, it still holds for t in the same range when t is replaced by $t - t_0$, that is

$$Q(D)y_1(t - t_0) = P(D)x_1(t - t_0), \qquad \text{for all } t,$$

so that

$$\text{(2.57)} \qquad Q(D)y_2(t) = P(D)x_2(t), \qquad \text{for all } t.$$

Also, from (2.55) and (2.56), we find

$$y_2(-\infty) = y_1(-\infty - t_0) = y_1(-\infty) = 0,$$

and, similarly for the higher-order derivatives of $y(t)$. Thus,

$$\text{(2.58)} \qquad y_2(-\infty) = 0, \quad y_2^{(1)}(-\infty) = 0, \ldots, \quad y_2^{(n-1)}(-\infty) = 0.$$

By (2.57) and (2.58) it follows that $y_2(t)$ is indeed the response of the system (2.51) to the input $x_2(t)$, as required to see time invariance. ∎

2.3.2 Zero-Input and Zero-State Response

The goal of this section is to determine the response of the system

$$(2.59)\qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

to an input signal $x(t)$ which is such that $x(t) = 0$ for all $t < 0$. To this end, let $y_{zi}(t)$ be the response of (2.59) to the zero input, that is, the response of the system

$$(2.60)\qquad \begin{cases} Q(D)y(t) = 0, \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

and let $y_{zs}(t)$ be the response of the system

$$(2.61)\qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = 0, \quad y^{(1)}(0-) = 0, \ldots, \quad y^{(n-1)}(0-) = 0. \end{cases}$$

Put

$$(2.62)\qquad y(t) \triangleq y_{zi}(t) + y_{zs}(t).$$

By linearity of derivatives we see that $y(t)$ given by (2.62) is indeed the response of (2.59), so that it remains to determine the signals $y_{zi}(t)$ and $y_{zs}(t)$ in order to find this response. We call the signal $y_{zi}(t)$ the **zero-input response** of (2.59), and call the signal $y_{zs}(t)$ the **zero-state response** of the system (2.59). We next consider how to find the zero-input and zero-state responses of (2.59):

2.3.3 The Zero-Input Response

Motivated by (2.42) define

$$(2.63)\qquad Q(\lambda) \triangleq \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0,$$

for all complex λ. We call $Q(\lambda)$ a polynomial of order n in the complex variable λ. For these polynomials there is a famous result called the **fundamental theorem of algebra** which goes as follows:

Theorem 2.3.6 *The polynomial $Q(\lambda)$ in (2.63) can be factorized as*

$$(2.64)\qquad Q(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_r)^{m_r},$$

where λ_i are complex constants and m_i are positive integers such that

$$n = m_1 + m_2 + \ldots + m_r. \tag{2.65}$$

The complex constants λ_i are called the **roots** of $Q(\lambda)$, and m_i is called the **multiplicity** of the root λ_i. Using the roots of $Q(\lambda)$ we are now going to *guess* a possible solution $y_{zi}(t)$ of (2.60). We consider two separate cases:

Case 1: The roots λ_i are distinct. This means that $r = n$ and $m_1 = m_2 = \ldots m_n = 1$ in (2.65). Now put

$$y_{zi}(t) \triangleq \sum_{i=1}^{n} c_i e^{\lambda_i t}, \tag{2.66}$$

where $c_1, c_2, \ldots, c_n$ are complex constants to be determined. First we show that

$$Q(D) y_{zi}(t) = 0. \tag{2.67}$$

To this end put

$$\phi_i(t) \triangleq e^{\lambda_i t}.$$

Using the notation in (2.41) we have

$$\psi_i(t) \triangleq (D - \lambda_i)\phi_i(t) = \frac{d\phi_i(t)}{dt} - \lambda_i \phi_i(t) = 0, \qquad \text{for all } t. \tag{2.68}$$

Now, from (2.64). we get

$$Q(D) = \left[\prod_{k=1, k\neq i}^{n} (D - \lambda_k)\right](D - \lambda_i),$$

so, from (2.68),

$$\begin{aligned} Q(D)\phi_i(t) &= \left[\prod_{k=1, k\neq i}^{n} (D - \lambda_k)\right](D - \lambda_i)\,\phi_i(t) \\ &= \left[\prod_{k=1, k\neq i}^{n} (D - \lambda_k)\right]\psi_i(t) \\ &= 0 \qquad \text{for all } t. \end{aligned} \tag{2.69}$$

From (2.66),

$$y_{zi}(t) = \sum_{i=1}^{n} c_i \phi_i(t),$$

hence using (2.69),

$$Q(D)y_{zi}(t) = \sum_{i=1}^{n} c_i Q(D)\phi_i(t) = 0,$$

as required to get (2.67). Notice that (2.67) holds regardless of the value of the constants c_i. These must now be chosen to satisfy the auxiliary condition of (2.60), namely

$$y_{zi}(0-) = \alpha_0, \quad y_{zi}^{(1)}(0-) = \alpha_1, \ldots, \quad y_{zi}^{(n-1)}(-0) = \alpha_{n-1}. \tag{2.70}$$

From (2.66) we have

$$y_{zi}^{(i)}(t) = c_1\lambda_1^i e^{\lambda_1 t} + c_2\lambda_2^i e^{\lambda_2 t} + \ldots + c_n\lambda_n^i e^{\lambda_n t},$$

for all $i = 0, 1, \ldots, n$, and taking $t = 0-$ then gives

$$y_{zi}^{(i)}(0-) = c_1\lambda_1^i + c_2\lambda_2^i + \ldots + c_n\lambda_n^i,$$

for all $i = 0, 1, \ldots, n$. Thus the conditions in (2.70) can be written as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \lambda_1^3 & \lambda_2^3 & \cdots & \lambda_n^3 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_n^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}, \tag{2.71}$$

which gives us n linear algebraic equations in the n unknowns $c_1, c_2, \ldots, c_n$. The matrix on the left hand side of (2.71) has an obviously special structure, and is called a **Vandermonde matrix**. Although we shall not do so here, it can be shown that the Vandermonde matrix is nonsingular when the complex numbers λ_i are distinct, as is being supposed. Thus, upon solving (2.71) for the constants c_i, we can substitute these into (2.66) to get the complete zero-input response.

Case 2: Some roots λ_i are repeated. This means that $m_i > 1$ for some $i = 1, 2, \ldots, r$ in Theorem 2.3.6. Now put

$$y_{zi}(t) \triangleq \sum_{i=1}^{r} \sum_{j=1}^{m_i} c_{i,j} t^{j-1} e^{\lambda_i t}, \tag{2.72}$$

where the $c_{i,j}$ are complex constants to be determined. Notice that (2.65) ensures that there are exactly n such constants. First we show that

$$Q(D)y_{zi}(t) = 0. \tag{2.73}$$

Put

$$\phi_{i,j}(t) \triangleq t^{j-1}e^{\lambda_i t},$$

for all $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, m_i$. Using the notation in (2.41) and an easy but tedious calculation we see that

$$\psi_{i,j}(t) \triangleq (D - \lambda_i)^{m_i}\phi_{i,j}(t) = 0, \qquad \text{for all } t, \tag{2.74}$$

for each $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, m_i$. Now, from (2.64). we get

$$Q(D) = \left[\prod_{k=1, k\neq i}^{n} (D - \lambda_k)^{m_k}\right](D - \lambda_i)^{m_i},$$

so that, from (2.74),

$$\begin{aligned} Q(D)\phi_{i,j}(t) &= \left[\prod_{k=1, k\neq i}^{r} (D - \lambda_k)^{m_k}\right](D - \lambda_i)^{m_i}\,\phi_{i,j}(t) \\ &= \left[\prod_{k=1, k\neq i}^{n} (D - \lambda_k)^{m_k}\right]\psi_{i,j}(t) \\ &= 0 \qquad \text{for all } t. \end{aligned} \tag{2.75}$$

Now, from (2.72), we have

$$y_{zi}(t) = \sum_{i=1}^{r}\sum_{j=1}^{m_i} c_{i,j}\phi_{i,j}(t),$$

hence using (2.75) we obtain

$$Q(D)y_{zi}(t) = \sum_{i=1}^{r}\sum_{j=1}^{m_i} c_{i,j}Q(D)\phi_{i,j}(t) = 0,$$

and (2.73) follows as required. It remains to find the constants $c_{i,j}$ in (2.72) such that

$$y_{zi}(0-) = \alpha_0, \quad y_{zi}^{(1)}(0-) = \alpha_1, \quad \ldots, \quad y_{zi}^{(n-1)}(0-) = \alpha_{n-1}. \tag{2.76}$$

To this end we take $(n-1)$-successive derivatives of (2.72) and use the conditions of (2.76) to get n-linear algebraic equations in the n-unknowns $c_{i,j}$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, m_i$.

2.3.4 The Zero-State Response

Here we shall determine the response $y_{zs}(t)$ of the system (2.61), namely

$$\begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = 0, \quad y^{(1)}(0-) = 0, \ldots, \quad y^{(n-1)}(0-) = 0, \end{cases} \tag{2.77}$$

where $x(t)$ is an input signal with $x(t) = 0$ for all $t < 0$. To this end, consider the supplementary system

$$\text{(2.78)} \qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(-\infty) = 0, \quad y^{(1)}(-\infty) = 0, \ldots, \quad y^{(n-1)}(-\infty) = 0, \end{cases}$$

which is similar to (2.77) except that the auxiliary conditions are now applied at $-\infty$, that is the signal $y(t)$ satisfying (2.78) is initially at rest. Since $x(t) = 0$ for all $t < 0$ we know from Remark 2.3.4 that $y(t) = 0$ for all $t < 0$, so that in particular we have

$$y(0-) = 0, \quad y^{(1)}(0-) = 0, \ldots, \quad y^{(n-1)}(0-) = 0.$$

It follows that, if $y(t)$ is the response of the system (2.78) to the input signal $x(t)$ with $x(t) = 0$ for all $t < 0$, then it is also the response of the system (2.77) to the same input signal $x(t)$. The useful thing about (2.78) is that it is an LTI and causal system, by Theorem 2.3.5, hence by Theorem 2.2.8 the response of (2.78) and therefore also the response of (2.77) is given by

$$\text{(2.79)} \qquad y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau,$$

where $h(\tau)$ is the impulse response of the LTI causal system (2.78). Now $x(\tau) = 0$ for all $\tau < 0$, so that (2.79) becomes

$$\text{(2.80)} \qquad y_{zs}(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Moreover, since (2.78) is causal we know that $h(u) = 0$ for all $u < 0$ (recall Section 2.2.1), thus $h(t-\tau) = 0$ for all $\tau > t$, and so (2.80) becomes

$$\text{(2.81)} \qquad y_{zs}(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau,$$

which is the desired zero-state response.

Remark 2.3.7 The question arises: how do we determine the impulse response $h(t)$ of the LTI causal system (2.78) so that we can evaluate (2.81)? In the next chapter we shall learn a powerful computational technique, based on the method of Laplace transforms, which enables us to easily find the impulse response as well as evaluate the convolution integral in (2.81). The method of Laplace transforms will also furnish a powerful alternative technique to the methods given here for determining the zero-input response whether or not any λ_i are repeated.

2.3.5 Stability

Consider the following system, where as usual the input signal $x(t)$ is such that $x(t) = 0$ for all $t < 0$.

$$(2.85) \qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{cases}$$

The *stability* of the system concerns the behaviour of the output $y(t)$ as $t \to \infty$. As we saw in Section 2.3.2 the response of this system is determined by its *zero-input response* $y_{zi}(t)$ and its *zero-state response* $y_{zs}(t)$. Thus, if we can identify long-run behaviour (as $t \to \infty$) for both the zero-input response $y_{zi}(t)$ and the zero-state response $y_{zs}(t)$, we will then be able to transfer this into knowledge about the long-run behaviour of the output $y(t)$, since

$$y(t) = y_{zi}(t) + y_{zs}(t).$$

Motivated by these considerations, we are going to formulate desirable long-run behaviour (called "stability") for both the zero-input response and the zero-state response of (2.85). We begin with the zero-input response:

Definition 2.3.7 The system (2.85) is called **asymptotically stable** when its zero-input response $y_{zi}(t)$ has the property that

$$\lim_{t \to \infty} y_{zi}(t) = 0$$

for each and every choice of constants $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$.

The intuitive idea of asymptotic stability is the following: when the input signal is the zero signal then the output signal $y(t)$ of (2.85), which is just the zero-input response $y_{zi}(t)$, tends towards the zero value, regardless of the initial conditions α_0, $\alpha_1, \ldots, \alpha_{n-1}$. How can we determine when (2.85) is asymptotically stable? As the next result shows, the essential determining element is the location of the roots λ_i of the polynomial $Q(\lambda)$ in (2.66) (recall Theorem 2.3.5):

Theorem 2.3.8 The system (2.85) is asymptotically stable if and only if

$$\mathrm{re}(\lambda_i) < 0 \qquad \text{for all } r = 1, 2, \ldots, r.$$

Remark 2.3.9 The theorem holds regardless of whether or not the roots of $Q(\lambda)$ are repeated.

Next, consider the long run behaviour of the zero-state response. As in Section 2.3.2 we are only interested in input signals which are such that $x(t) = 0$ for all $t < 0$. The desirable long-run behaviour that we want to define precisely is that a *bounded* input signal $x(t)$ should give rise to only a *bounded* zero-state response. To do this, we must first make the notion of a bounded signal precise:

Definition 2.3.10 A signal $x(t)$ with $x(t) = 0$ for all $t < 0$ is called *bounded* when there is some *finite* number $B > 0$ such that

$$|x(t)| < B, \qquad \text{for all } t \geq 0.$$

Now we can formulate stability of the zero-state response:

Definition 2.3.11 The system (2.85) is called **bounded input bounded output** or *BIBO* stable when every bounded input signal causes a bounded zero-state response $y_{zs}(t)$.

Theorem 2.3.12 The system (2.85) is BIBO stable if and only if $m \leq n$ (see (2.39)) and

$$\text{re}(\lambda_i) < 0 \qquad \text{for all } r = 1, 2, \ldots, r.$$