Chapter 6

Non-Periodic Inputs and Fourier Transforms

In Chapter 5 we learned how to use Fourier series to determine the response of a continuous-time system

(6.1) $$Q(D)y(t) = P(D)x(t),$$

for *periodic* input signals $x(t)$. There are many applications, especially in communications engineering, where it is important to determine the response of this system when the input signal $x(t)$ is neither periodic, as was postulated in Chapter 5, nor satisfies the right-sided condition

(6.2) $$x(t) = 0, \qquad \text{for all } t < 0,$$

which is crucial to the application of the Laplace transform method learned in Chapter 3. Indeed, in communications engineering one often uses systems of the form (6.1) to "process" non-periodic input signals which represent very long strings of information going indefinitely far back into the distant past. Such signals must be allowed to have non-zero values for arbitrarily large negative values of t, and the condition (6.2) is therefore clearly inappropriate. In determining the response of (6.1) we are thus excluded from using both Fourier series (since the signal is generally not periodic) and Laplace transforms (since (6.2) does not hold). Our goal in this chapter is to learn yet another mathematical technique, which also originates in the profound works of Fourier, namely *Fourier transforms*, which is ideally suited to this problem.

6.1 From Fourier Series to Fourier Transforms

Let $x(t)$ be a complex-valued signal such that, for some finite $A > 0$ we have

(6.3) $$x(t) = 0, \qquad \text{for all } |t| > A.$$

Such a signal is usually called **time-bounded** since it is nonzero only over the restricted time interval $[-A, A]$. See Fig. 6.1.

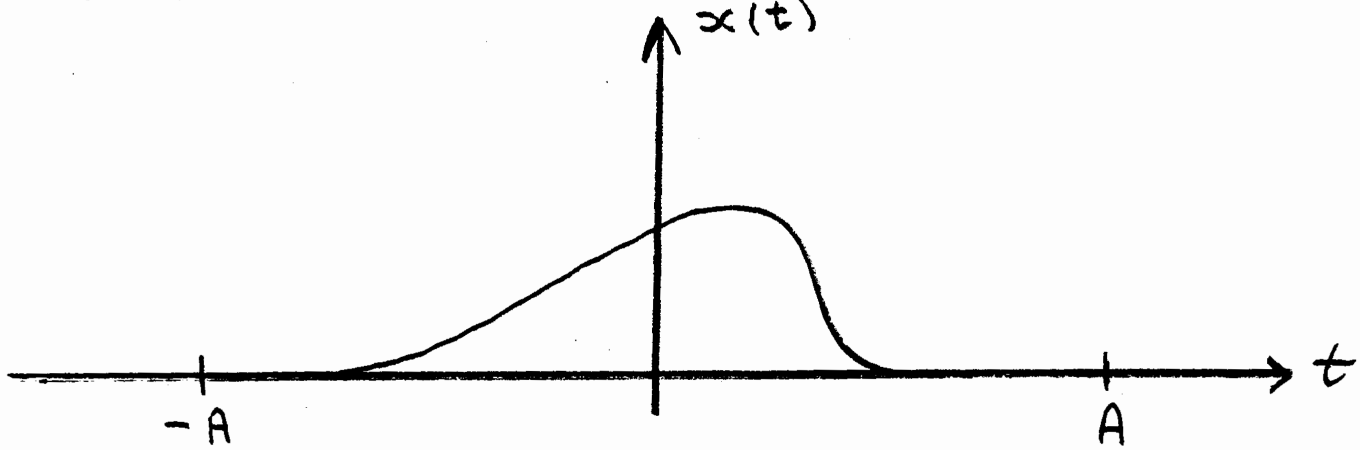


Fig. 6.1

Now fix some $T > 2A$, and define the **periodic extension** $x_T(t)$ of $x(t)$ to be the periodic signal with period T such that

(6.4) $$x_T(t) = x(t), \qquad \text{for all } |t| < T/2$$

See Fig. 6.2.

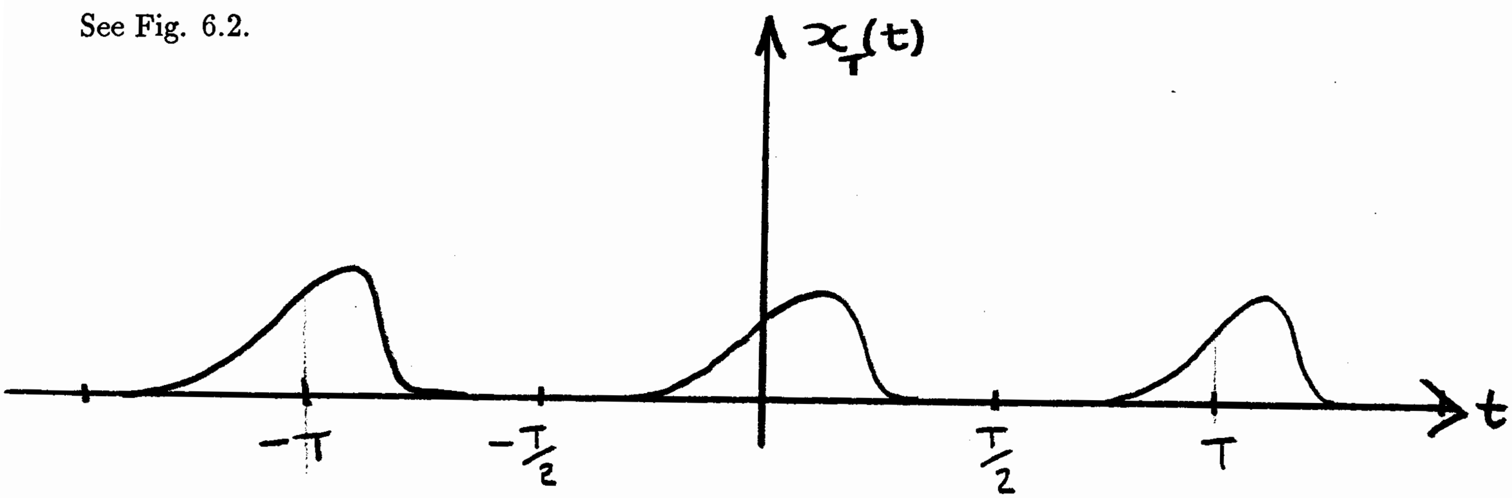


Fig. 6.2

One easily observes that, for each fixed t we have

(6.5) $$\lim_{T\to\infty} x_T(t) = x(t).$$

We now take an exponential Fourier series expansion of the periodic signal $x_T(t)$, namely

$$x_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{6.6}$$

with, as usual,

$$\omega_0 \triangleq \frac{2\pi}{T} \tag{6.7}$$

and

$$a_k = \frac{1}{T}\int_0^T x_T(t)e^{-jk\omega_0 t}\,dt.$$

Now $x_T(t)$ is, by its very definition, periodic with period T, while $e^{jk\omega_0 t}$ is clearly periodic, also with a period of T, so that $x_T(t)e^{jk\omega_0 t}$ is periodic with a period T. Thus we can use Remark 1.1.11 with $t_0 = -T/2$ to write a_k in the form

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}\,dt. \tag{6.8}$$

Putting (6.6), (6.7), and (6.8) together then yields

$$\begin{aligned} x_T(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{T}\left[\int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}\,dt\right] e^{jk\omega_0 t} \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}\,dt\right]\omega_0 e^{jk\omega_0 t}. \end{aligned} \tag{6.9}$$

Since $T > 2A$, we see from (6.3) that

$$x(t) = \begin{cases} x_T(t), & \text{for all } |t| \le T/2, \\ 0, & \text{for all } |t| > T/2. \end{cases}$$

Thus

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,dt &= \int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt \\ &= \int_{-T/2}^{T/2} x_T(t)e^{-jk\omega_0 t}\,dt. \end{aligned} \tag{6.10}$$

Combining (6.9) and (6.10) gives

$$x_T(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,dt\right]\omega_0 e^{jk\omega_0 t}. \tag{6.11}$$

Now define

$$X(\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \qquad \text{for all } -\infty < \omega < \infty,$$

so that

$$X(k\omega_0) = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\, dt. \tag{6.12}$$

From (6.11) and (6.12) we see that

$$x_T(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0)\omega_0 e^{jk\omega_0 t}, \tag{6.13}$$

and, from (6.7), we see that

$$T \to \infty \qquad \text{if and only if} \qquad \omega_0 \to 0.$$

It follows that

$$\begin{aligned} \lim_{T\to\infty} \left\{ \sum_{k=-\infty}^{\infty} X(k\omega_0)\omega_0 e^{jk\omega_0 t} \right\} &= \lim_{\omega_0\to 0} \left\{ \sum_{k=-\infty}^{\infty} X(k\omega_0)\omega_0 e^{jk\omega_0 t} \right\} \\ &= \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega. \end{aligned} \tag{6.14}$$

From (6.5), (6.13) and (6.14) we see that

$$\begin{aligned} x(t) &= \lim_{T\to\infty} x_T(t) \\ &= \lim_{T\to\infty} \left\{ \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0)\omega_0 e^{jk\omega_0 t} \right\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} dt. \end{aligned}$$

These manipulations suggest (but do not prove!) the following extremely important result known as **Fourier's theorem:**

Theorem 6.1.1 Let $x(t)$ be a complex-valued signal and define

$$X(\omega) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt, \qquad \text{for all } -\infty < \omega < \infty. \tag{6.15}$$

Then

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega, \qquad \text{for all instants } -\infty < t < \infty, \tag{6.16}$$

provided the integrals in (6.15) and (6.16) make sense.

Remark 6.1.2 The function $X(\omega)$ of $-\infty < \omega < \infty$, defined in terms of the signal $x(t)$ by (6.15), is called the **Fourier transform** of the signal $x(t)$, and is also denoted by

$$\mathcal{F}\{x\}(\omega) \qquad \text{or} \qquad \mathcal{F}\{x(t)\}(\omega).$$

Likewise, the signal $x(t)$ derived from the given function $X(\omega)$ by (6.16) is called the **inverse Fourier transform** of $X(\omega)$, and is denoted by

$$\mathcal{F}^{-1}\{X(\omega)\}(t).$$

Remark 6.1.3 We saw in Chapter 5 how periodic signals can be decomposed into a sum of exponential sinusoids. It was the inspired insight of Fourier in the early 19-th century that a *non-periodic* signal $x(t)$ should similarly be decomposable into a *generalized sum* (or integral) of exponential sinusoids, and this is exactly what Theorem 6.1.1 shows: From (6.16) we see that $x(t)$ is a generalized sum of the exponential sinusoids

$$e^{j\omega t}$$

added up over *all frequencies* $-\infty < \omega < \infty$, with "weight" at each frequency ω given by $X(\omega)$. This should be contrasted with the situation for periodic signals, which can be decomposed into a sum of the exponential sinusoids over a *discrete set* of frequencies

$$\omega = k\omega_0, \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots$$

with "weight" at frequency $k\omega_0$ given by the Fourier coefficient a_k. The manipulations that we have used to motivate the relations (6.15) and (6.16), as well as the statement of Theorem 6.1.1, follow almost exactly the 1807 work of Fourier "*Théorie analytique de la chaleur*" (The analytical theory of heat), where these ideas were presented for the first time. It is well to note that the preceding manipulations make Theorem 6.1.1 a very plausible result, but are not by any means a watertight proof. Indeed, it is exceedingly difficult to give a completely rigorous proof of Theorem 6.1.1, and it was not until the late 1950's that this was accomplished in the pioneering work of Laurent Schwartz "*Théorie des distributions*", 1957, 1959, which make use of mathematical tools and ideas that simply did not exist in Fourier's time. Needless to say, we do not attempt to prove Theorem 6.1.1 here. In fact our attitude towards this theorem will be very similar to that of Fourier himself, who was primarily interested in using it as a tool for studying problems in mathematical physics (the distribution and propagation of heat in solids). In much the same spirit we are going to regard Theorem 6.1.1 as a tool for dealing with problems in system theory.

Remark 6.1.4 The function $X(\omega)$ of ω given by (6.15) is also called the **frequency spectrum** of the signal $x(t)$, and is a generalization to non-periodic signals of the frequency spectrum a_k for periodic signals $x(t)$ (recall Remark 5.1.2). Generally $X(\omega)$ is a complex-valued function of ω so that it must be displayed in two graphs, namely a **magnitude plot** of $|X(\omega)|$ versus ω and a **phase plot** of $\angle X(\omega)$ versus ω.

Example 6.1.5 Determine the Fourier transform of the signal

$$x(t) = e^{-at}u(t),$$

where $a > 0$ is a real constant. From (6.15) we have

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

The magnitude and phase plots of the frequency spectrum are shown in Fig. 6.3.

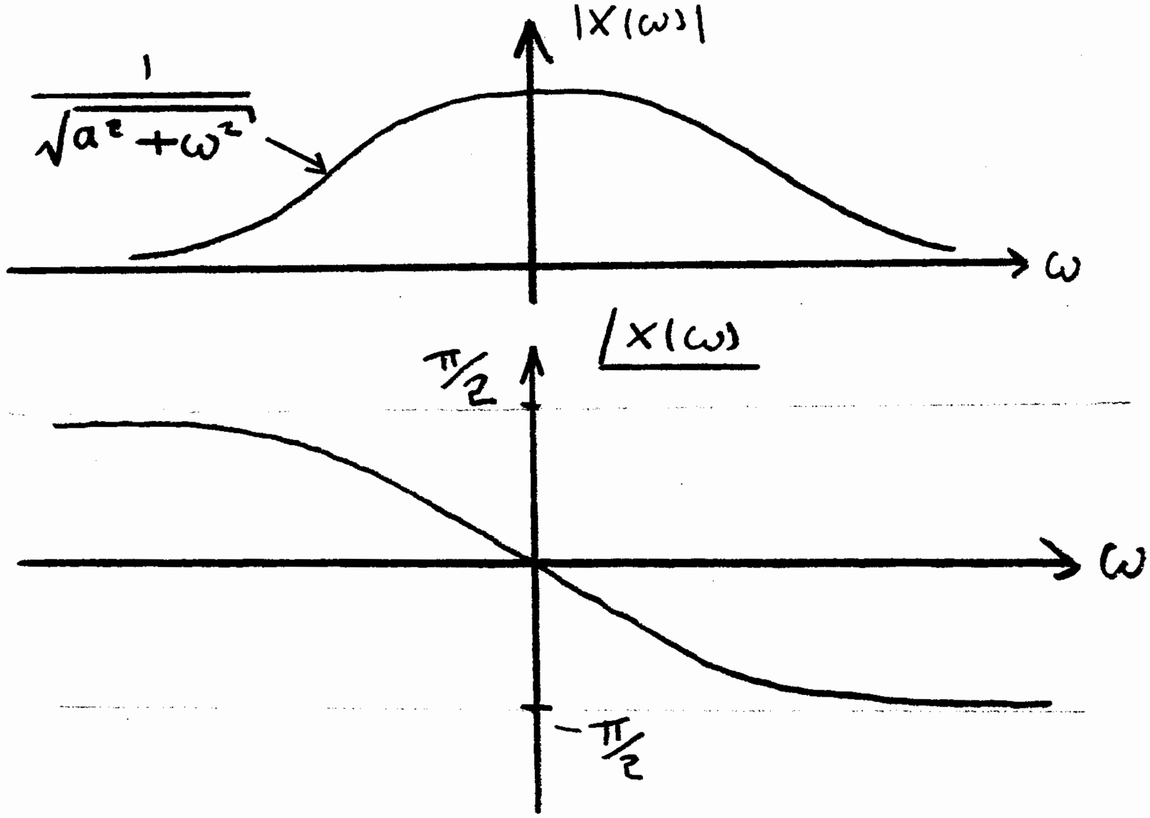


Fig. 6.3

Remark 6.1.6 Real-valued signals $x(t)$ are particularly important in applications. In this case we clearly have

$$x(t)e^{j\omega t} = [x(t)e^{-j\omega t}]^*,$$

thus (replacing ω by $-\omega$ in (6.15) we see

$$\begin{aligned} X(-\omega) &= \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt \\ &= \int_{-\infty}^{\infty} [x(t)e^{-j\omega t}]^* dt \\ &= \left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right\}^* \\ &= X^*(\omega). \end{aligned}$$

To summarize, when $x(t)$ is a real-valued signal we have

$$X(-\omega) = X^*(\omega), \qquad \text{or equivalently} \qquad X(\omega) = X^*(-\omega), \qquad \text{for all } \omega, \tag{6.17}$$

and from this we see that

$$|X(\omega)| = |X^*(-\omega)| = |X(-\omega)|, \qquad \text{for all } \omega,$$

so that the magnitude plot is an *even function* of ω, and

$$\angle X(\omega) = \angle X^*(-\omega) = -\angle X(-\omega), \qquad \text{for all } \omega, \tag{6.18}$$

so that the phase plot is an *odd function* of ω. This is very similar to the property of Fourier coefficients of real signals noted in Remark 5.1.3.

Remark 6.1.7 Now suppose that the signal $x(t)$ is both real-valued and an *even* function, so that

$$x(t) = x(-t).$$

From (6.15) we have

$$\begin{aligned} X(-\omega) &= \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(-\tau)e^{-j\omega\tau}d\tau \qquad (\text{with } \tau = -t), \\ &= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \qquad (\text{since } x(\tau) = x(-\tau)), \\ &= X(\omega). \end{aligned} \tag{6.19}$$

It follows that $X(\omega)$ is an **even function** of ω, and using (6.17), we find

$$X(\omega) = X(-\omega) = X^*(\omega),$$

which shows that $X(\omega)$ is also a **real-valued** function of ω. In the same way, if $x(t)$ is a **real-valued** and **odd function** of t, then $X(\omega)$ is an **imaginary-valued** and **odd function** of ω.

6.2 Properties of Fourier Transforms

We list the most important properties of Fourier transforms. These properties are established by simple manipulations of (6.15) and (6.16). We omit the proofs since our main interest is in the use of these properties.

(I) LINEARITY: Suppose that $x_1(t)$, $x_2(t)$, are signals and c_1, c_2, are complex constants. Then

$$\mathcal{F}\{c_1x_1 + c_2x_2\}(\omega) = c_1\mathcal{F}\{x_1\}(\omega) + c_2\mathcal{F}\{x_2\}(\omega).$$

(II) SYMMETRY PROPERTIES: Suppose that $x(t)$ is a real-valued signal with Fourier transform $X(\omega)$. Then

$$X(-\omega) = X^*(\omega).$$

(III) TIME SHIFT: If $x(t)$ is a signal with Fourier transform $X(\omega)$ and t_0 is a fixed instant, then

$$\mathcal{F}\{x(t-t_0)\}(\omega) = e^{-j\omega t_0}X(\omega).$$

(IV) FREQUENCY SHIFT: If $x(t)$ is a signal with Fourier transform $X(\omega)$ and ω_0 is a fixed frequency, then

$$\begin{aligned}
\mathcal{F}\left\{e^{j\omega_0 t}x(t)\right\} &= X(\omega-\omega_0); \\
\mathcal{F}\{\cos(\omega_0 t)x(t)\}(\omega) &= \frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]; \\
\mathcal{F}\{\sin(\omega_0 t)x(t)\}(\omega) &= -\frac{j}{2}\left[X(\omega-\omega_0) - X(\omega+\omega_0)\right].
\end{aligned}$$

(V) DERIVATIVES: For a signal $x(t)$ with Fourier transform $X(\omega)$ we have

$$\mathcal{F}\left\{x^{(n)}(t)\right\}(\omega) = (j\omega)^n X(\omega), \qquad n = 1, 2, 3, \ldots$$

(VI) INTEGRALS: If a signal $x(t)$ has Fourier transform $X(\omega)$ and

$$y(t) \triangleq \int_{-\infty}^{t} x(\tau)d\tau,$$

then

$$\mathcal{F}\{y(t)\}(\omega) = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega).$$

(VII) TIME SCALING: If a signal $x(t)$ has Fourier transform $X(\omega)$ and a is a real constant then

$$\mathcal{F}\{x(at)\}(\omega) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right).$$

(VIII) DUALITY: If $x(t)$ is a signal with Fourier transform $X(\omega)$, then

$$\mathcal{F}\{X(t)\}(\omega) = 2\pi x(-\omega).$$

(IX) CONVOLUTION: Suppose that $x(t)$, $y(t)$, are signals with Fourier transforms $X(\omega)$, $Y(\omega)$, respectively. Then

$$\mathcal{F}\{x(t) * y(t)\}(\omega) = X(\omega)Y(\omega).$$

(X) MULTIPLICATION: Suppose that $x(t)$, $y(t)$, are signals with Fourier transforms $X(\omega)$, $Y(\omega)$, respectively. Then

$$\begin{aligned}
\mathcal{F}\{x(t)y(t)\}(\omega) &= \frac{1}{2\pi}(X * Y)(\omega) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)Y(\omega-\lambda)d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda)X(\omega-\lambda)d\lambda.
\end{aligned}$$

Example 6.2.1 Determine Fourier transforms for the signals

(i) $x(t) = e^{-|t|}$;

(ii) $y(t) = \frac{2}{t^2+1}$.

(i) We have

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \left[\int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt\right] \\
&= \left[\frac{1}{1-j\omega} + \frac{1}{1+j\omega}\right] \\
&= \frac{2}{1+\omega^2}.
\end{aligned}$$

(ii) From the Fourier transform in (i) we see that

$$y(t) = X(t).$$

By duality (Property VIII) we see that

$$\begin{aligned}
\mathcal{F}\{y(t)\}(\omega) &= \mathcal{F}\{X(t)\}(\omega) \\
&= 2\pi x(-\omega) \\
&= 2\pi e^{-|\omega|}.
\end{aligned}$$

Example 6.2.2 Determine Fourier transforms of the signals

$$x(t) = \begin{cases} 1, & \text{for all } -T \leq t \leq T, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$y(t) = \frac{1}{\pi t}\sin(Wt),$$

where $T, W > 0$ are constants.

For the signal $x(t)$:

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-T}^{T} e^{-j\omega t}dt \\
&= \left.\frac{-1}{j\omega}e^{-j\omega t}\right|_{-T}^{T} \\
&= \frac{2}{\omega}\sin(\omega T).
\end{aligned}$$

Next consider the signal $y(t)$. Using the Fourier transform $X(\omega)$ duality (Property (VIII)) we get

$$\begin{aligned}
\mathcal{F}\left\{\frac{2}{t}\sin(tT)\right\}(\omega) &= \mathcal{F}\{X(t)\}(\omega) \\
&= 2\pi x(-\omega) \\
&= 2\pi x(\omega) \\
&= \begin{cases} 2\pi, & \text{for all } -T \leq \omega \leq T, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, replacing T with W,

$$\mathcal{F}\left\{\frac{2}{t}\sin(Wt)\right\}(\omega) = \begin{cases} 2\pi, & \text{for all } -W \leq \omega \leq W, \\ 0, & \text{otherwise.} \end{cases}$$

Thus we get

$$\begin{aligned} \mathcal{F}\left\{\frac{1}{\pi t}\sin(Wt)\right\}(\omega) &= \frac{1}{2\pi}\mathcal{F}\left\{\frac{2}{t}\sin(Wt)\right\}(\omega) \\ &= \begin{cases} 1, & \text{for all } -W \leq \omega \leq W, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Remark 6.2.3 Motivated by the signal $x(t)$ in Example 6.2.2, we define the **rectangular pulse function of unit width** as follows:

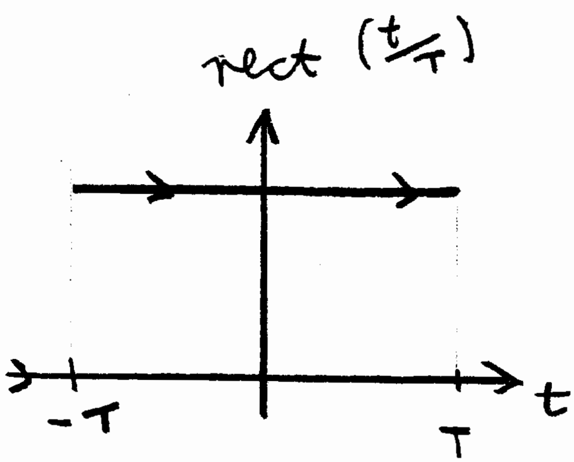
$$\operatorname{rect}(t) = \begin{cases} 1, & \text{for all } -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also define the so-called **sinc function** $\operatorname{sinc}(x)$ as follows:

$$\operatorname{sinc}(x) = \frac{1}{\pi x}\sin(\pi x), \qquad \text{for all real } x.$$

Then from Example 6.2.2 we see that

$$\begin{aligned} \mathcal{F}\left\{\operatorname{rect}\left(\frac{t}{T}\right)\right\}(\omega) &= \frac{2}{\omega}\sin(\omega T) \\ &= \frac{2T\sin(\pi(\omega T/\pi))}{\pi(\omega T/\pi)} \\ &= 2T\operatorname{sinc}\left(\frac{\omega T}{\pi}\right). \end{aligned}$$



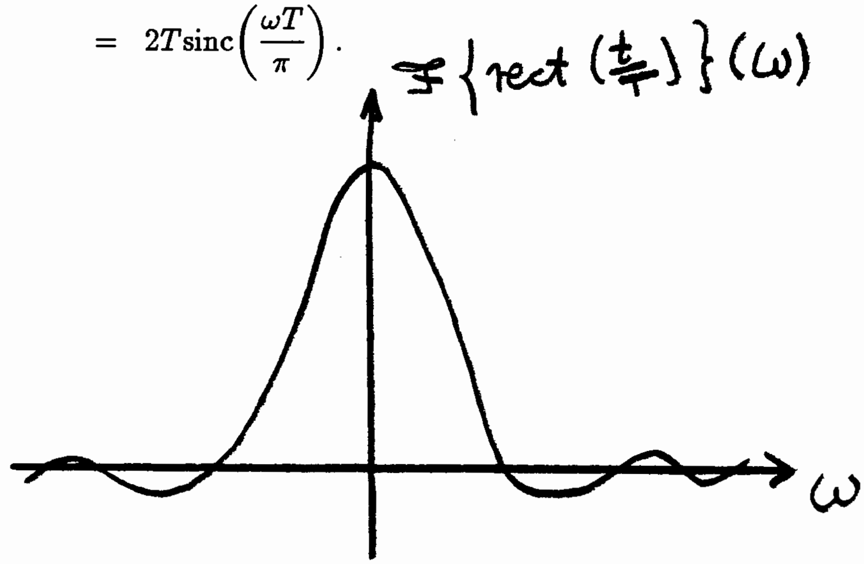


Fig. 6.4

Example 6.2.4 Determine $\mathcal{F}\{\delta(t)\}(\omega)$ and then show that

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}d\omega.$$

From (6.15) we have

$$\begin{aligned} X(\omega) &= \mathcal{F}\{\delta(t)\}(\omega) \\ &= \int_{-\infty}^{\infty} \delta(\tau)e^{-j\omega\tau}d\tau \\ &= 1. \end{aligned} \tag{6.20}$$

Then, using Theorem 6.1.1, we find

$$\begin{aligned} \delta(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega\tau}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}d\omega. \end{aligned}$$

Example 6.2.5 Determine $\mathcal{F}\{\delta(t-t_0)\}(\omega)$ where t_0 is a fixed instant.

If we put $X(\omega) \triangleq \mathcal{F}\{\delta(t)\}(\omega)$ then, by the time-shift property (Property (III)), we find

$$\mathcal{F}\{\delta(t-t_0)\}(\omega) = e^{-j\omega t_0}X(\omega).$$

But, in Example 6.2.4 we show that

$$X(\omega) = 1,$$

hence

$$\mathcal{F}\{\delta(t-t_0)\}(\omega) = e^{-j\omega t_0}.$$

Example 6.2.6 Determine $\mathcal{F}\{u(t)\}(\omega)$.

We know that

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau.$$

Put $X(\omega) = \mathcal{F}\{\delta(t)\}(\omega)$. By Property (VI) we see

$$\begin{aligned} \mathcal{F}\{u(t)\}(\omega) &= \mathcal{F}\left\{\int_{-\infty}^{t} \delta(\tau)d\tau\right\}(\omega) \\ &= \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \end{aligned}$$

From Example 6.2.4 we know that

$$X(\omega) = 1$$

giving

$$\mathcal{F}\{u(t)\}(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Example 6.2.7 Determine the Fourier transforms

(i) $\mathcal{F}\{1\}(\omega)$;

(ii) $\mathcal{F}\{e^{j\omega_0 t}\}(\omega)$;

(iii) $\mathcal{F}\{\cos(\omega_0 t)\}(\omega)$.

(i) In Example 6.2.4 we show

$$\mathcal{F}\{\delta(t)\}(\omega) = 1,$$

so that duality Property (VIII) gives

$$\begin{aligned}\mathcal{F}\{1\}(\omega) &= 2\pi\delta(-\omega)\\ &= 2\pi\delta(\omega).\end{aligned}$$

(ii) Put $x(t) = 1$. From (i) we see that its Fourier transform $X(\omega)$ is given by

$$X(\omega) = 2\pi\delta(\omega).$$

Using the frequency shift Property (IV) we get

$$\begin{aligned}\mathcal{F}\left\{e^{j\omega_0 t}\right\}(\omega) &= \mathcal{F}\left\{e^{j\omega_0 t}x(t)\right\}(\omega)\\ &= X(\omega - \omega_0)\\ &= 2\pi\delta(\omega - \omega_0). \qquad (6.21)\end{aligned}$$

(iii) From (ii) we have

$$\mathcal{F}\left\{e^{-j\omega_0 t}\right\}(\omega) = 2\pi\delta(\omega + \omega_0),$$

thus

$$\begin{aligned}\mathcal{F}\{\cos(\omega_0 t)\}(\omega) &= \frac{1}{2}\mathcal{F}\left\{e^{j\omega_0 t}\right\}(\omega) + \frac{1}{2}\mathcal{F}\left\{e^{-j\omega_0 t}\right\}(\omega)\\ &= \frac{1}{2}\left[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right]\\ &= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0).\end{aligned}$$

Example 6.2.8 Determine the Fourier transform of the signal

$$e^{at}u(-t),$$

where $a > 0$ is a constant.

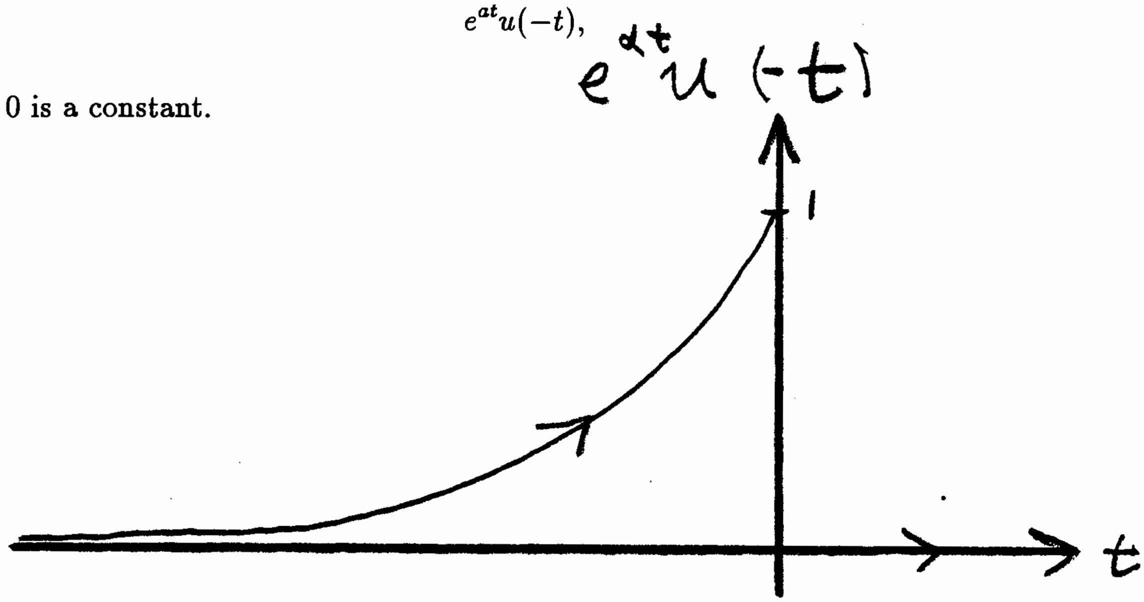


Fig. 6.5

From Example 6.1.5 we know

$$\mathcal{F}\{e^{-at}u(t)\}(\omega) = \frac{1}{a + j\omega}.$$

Then, from the symmetry Property (II) we get

$$\mathcal{F}\{e^{at}u(-t)\}(\omega) = \frac{1}{a - j\omega}.$$

In using Fourier transforms we shall often be in the situation where we have a given function $X(\omega)$, $-\infty < \omega < \infty$, and must determine its inverse Fourier transform (recall Remark 6.1.2), namely the signal $x(t)$ which is given by (6.16). Indeed, we shall encounter situations of exactly this sort when we look at the response of a system to a non-periodic input signal. Of course, one can in principle determine the inverse Fourier transform by evaluating the integral in (6.16), but in practice this is often a tedious procedure. Much as was the case for Laplace transforms and z-transforms we shall usually only want inverse Fourier transforms for *rational functions* of ω, which we can expand by the method of partial fractions. As a consequence of this reduction procedure, we usually require the inverse Fourier transforms of only a rather small class of functions $X(\omega)$, which we can easily collect in a table as follows:

	Signal $x(t)$	Fourier Transform $X(\omega)$
1.	1	$2\pi\delta(\omega)$
2.	$\delta(t)$	1
3.	$\delta^{(n)}(t)$, $n = 1, 2, \ldots$	$(j\omega)^n$
4.	$u(t)$	$1/(j\omega) + \pi\delta(\omega)$
5.	$t^n e^{-\alpha t}u(t)$, α complex, $\mathrm{re}(\alpha) > 0$, $n = 0, 1, 2, \ldots$	$n!/(\alpha + j\omega)^{n+1}$
6.	$(-t)^n e^{\alpha t}u(-t)$, α complex, $\mathrm{re}(\alpha) > 0$, $n = 0, 1, 2, \ldots$	$n!/(\alpha - j\omega)^{n+1}$
7.	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$

6.3 Signal Energy and the Parseval Theorem

Suppose that $x(t)$ is an **energy signal**, that is (recall Definition 1.1.15) for

$$E \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

we have $E < \infty$. Since $x(t)$ is complex-valued we see from (6.16) that

$$\begin{aligned} [x(t)]^* &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right]^* \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega. \end{aligned}$$

Thus

$$|x(t)|^2 = x(t)[x(t)]^* = \frac{x(t)}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega,$$

and therefore the total energy in the signal is given by

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \frac{x(t)}{2\pi}\left[\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega\right] dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)X^*(\omega)e^{-j\omega t} \, d\omega \, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)X^*(\omega)e^{-j\omega t}\,dt\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right]\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)X(\omega)\,d\omega,
\end{aligned}
$$

where (6.15) is used at the last line. We have established the following result called **Parseval's theorem:**

Theorem 6.3.1 Suppose that $x(t)$ is a signal with Fourier transform $X(\omega)$ such that

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt < \infty.$$

Then

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega.$$

For an angular frequency $\omega_B > 0$ we call

$$\frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} |X(\omega)|^2\,d\omega$$

the **total energy** in the **frequency band** $-\omega_B$ to ω_B.

Example 6.3.2 Consider the signal

$$x(t) = e^{-at}u(t),$$

where $a > 0$ is a constant. Determine the angular frequency ω_B such that one-half of the total energy in the signal is in the frequency band $-\omega_B$ to ω_B.

The total energy in the signal is

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} |x(t)|^2\,dt \\
&= \int_{0}^{\infty} e^{-2at}\,dt \\
&= \frac{1}{2a}. \qquad (6.22)
\end{aligned}
$$

Now, from Example 6.1.5, the Fourier transform of the signal $x(t)$ is

$$X(\omega) = \frac{1}{a + j\omega}.$$

Then the energy in the frequency band $-\omega_B$ to ω_B is given by

$$
\begin{aligned}
E_B &= \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} |X(\omega)|^2\, d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} \frac{1}{a^2+\omega^2}\, d\omega \\
&= \frac{1}{\pi}\int_{0}^{\omega_B} \frac{1}{a^2+\omega^2}\, d\omega \\
&= \frac{1}{a\pi}\arctan\left(\frac{\omega_B}{a}\right). \qquad (6.23)
\end{aligned}
$$

We must find ω_B such that

$$\frac{1}{a\pi}\arctan\left(\frac{\omega_B}{a}\right) = \frac{1}{4a},$$

thus $\omega_B = a$.

6.4 Systems and Non-Periodic Inputs

In Section 5.2 we learned how to determine the response of the initially-at-rest system

$$
(6.24) \qquad \begin{cases} Q(D)y(t) = P(D)x(t), \\ y(-\infty) = 0, \quad y^{(1)}(-\infty) = 0, \ldots, \quad y^{(n-1)}(-\infty) = 0, \end{cases}
$$

to a *periodic* input signal $x(t)$ using Fourier series. In this section we shall deal with the same problem, but no longer insisting that $x(t)$ be periodic. We must allow for signals $x(t)$ for which the right-sided condition (6.2) does not necessarily hold and hence Laplace transform methods, which depend critically on this condition, are not available to us. As in Section 5.2 we shall always suppose that (6.24) is BIBO-stable. The method for determining the response of the system (6.24) is best shown first by an example and then extended to the general case:

Example 6.4.1 Determine the response of the initially-at-rest system

$$
(6.25) \qquad \begin{cases} (D+2)y(t) = x(t), \\ y(-\infty) = 0, \end{cases}
$$

for the input signal

$$x(t) = e^{3t}u(-t).$$

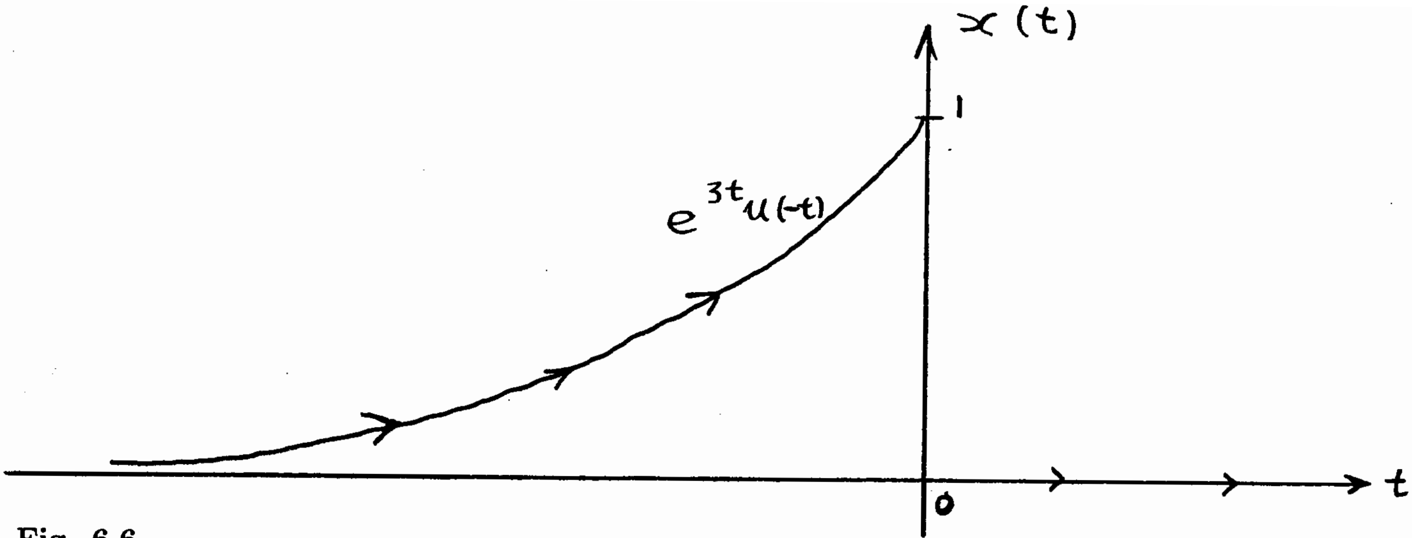


Fig. 6.6

Take Fourier transforms on each side of the differential equation:

$$\mathcal{F}\{Dy(t)\}(\omega) + 2\mathcal{F}\{y(t)\}(\omega) = \mathcal{F}\{x(t)\}(\omega). \tag{6.26}$$

Put

$$X(\omega) \triangleq \mathcal{F}\{x(t)\}(\omega), \qquad Y(\omega) \triangleq \mathcal{F}\{y(t)\}(\omega).$$

From Property (V) of derivatives,

$$\mathcal{F}\{Dy(t)\}(\omega) = j\omega Y(\omega). \tag{6.27}$$

We determine $X(\omega)$ next. From entry no 6. in the table of Fourier transform pairs (with $n = 0$, $\alpha = 3$) we find

$$X(\omega) = \mathcal{F}\{x(t)\}(\omega) = \frac{1}{3 - j\omega}. \tag{6.28}$$

From (6.26), (6.27) and (6.28) we get

$$j\omega Y(\omega) + 2Y(\omega) = \frac{1}{3 - j\omega},$$

and hence

$$Y(\omega) = \frac{1}{(3 - j\omega)(2 + j\omega)}.$$

(continues next page)

By the method of partial fractions we easily get

$$\frac{1}{(3-\vartheta)(2+\vartheta)} = \frac{1}{5}\left[\frac{1}{2+\vartheta} + \frac{1}{3-\vartheta}\right],$$

where ϑ is an arbitrary variable that we now take to be $\vartheta \triangleq j\omega$:

$$\begin{aligned} Y(\omega) &= \frac{1}{(3-j\omega)(2+j\omega)} \\ &= \frac{1}{5}\left[\frac{1}{2+j\omega} + \frac{1}{3-j\omega}\right]. \end{aligned} \tag{6.29}$$

From entry no 5. in the table of Fourier transforms (with $n = 0$ and $\alpha = 2$) we find

$$\mathcal{F}^{-1}\left\{\frac{1}{2+j\omega}\right\}(t) = e^{-2t}u(t). \tag{6.30}$$

From entry no 6. in the table of Fourier transforms (with $n = 0$ and $\alpha = 3$) we find

$$\mathcal{F}^{-1}\left\{\frac{1}{3-j\omega}\right\}(t) = e^{3t}u(-t). \tag{6.31}$$

From (6.29), (6.30), and (6.31):

$$y(t) = \frac{1}{5}\left[e^{-2t}u(t) + e^{3t}u(-t)\right].$$

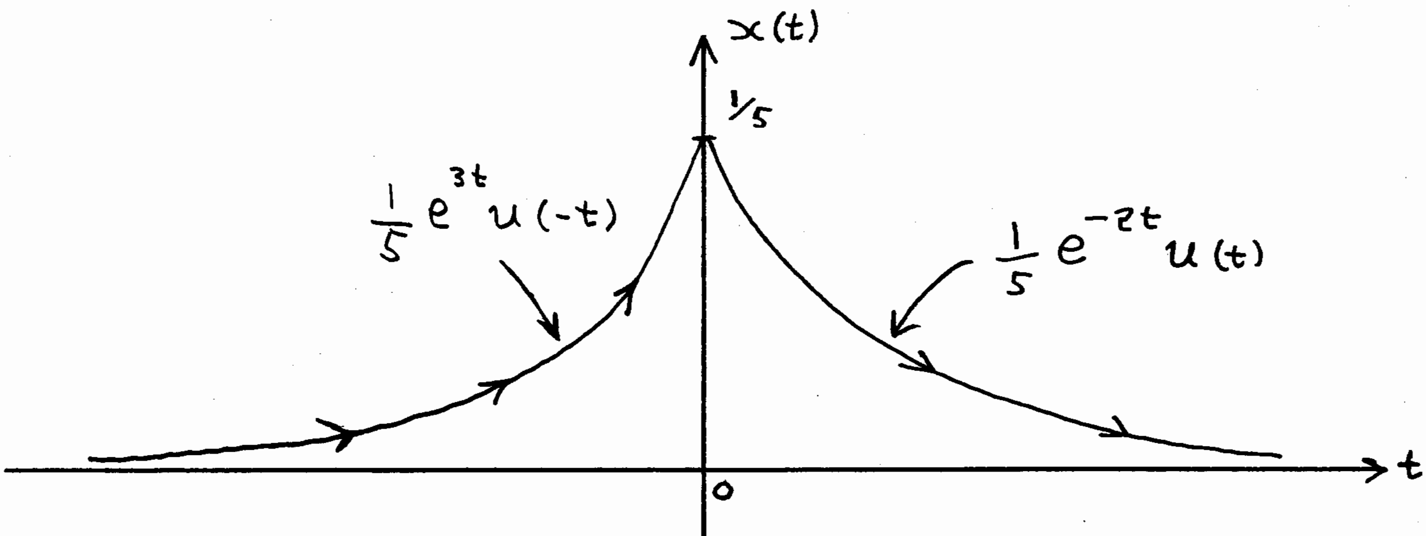


Fig. 6.7

We can repeat the method used for Example 6.4.1 in general to determine the Fourier transform of the response $y(t)$ of the system (6.24). Recall that

$$\begin{aligned} Q(D) &\triangleq D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0, \\ P(D) &\triangleq b_m D^m + b_{m-1}D^{m-1} + \ldots b_1 D + b_0, \end{aligned}$$

and put

$$X(\omega) \triangleq \mathcal{F}\{x(t)\}(\omega), \qquad\qquad Y(\omega) \triangleq \mathcal{F}\{y(t)\}(\omega).$$

From Property (V) of Fourier transforms we get

$$\mathcal{F}\{D^i y(t)\}(\omega) = (j\omega)^i Y(\omega),$$

from which it follows by linearity (Property (I)) that

$$\begin{aligned}\mathcal{F}\{Q(D)y(t)\}(\omega) &= \mathcal{F}\{D^n y(t)\}(\omega) + a_{n-1}\mathcal{F}\{D^{n-1}y(t)\}(\omega) + \ldots + a_1\mathcal{F}\{Dy(t)\}(\omega) + a_0\mathcal{F}\{y(t)\}(\omega) \\ &= (j\omega)^n Y(\omega) + a_{n-1}(j\omega)^{n-1}Y(\omega) + \ldots + a_1(j\omega)Y(\omega) + a_0 Y(\omega),\end{aligned}$$

and so

$$\mathcal{F}\{Q(D)y(t)\}(\omega) = Q(j\omega)Y(\omega). \tag{6.33}$$

Likewise,

$$\mathcal{F}\{P(D)x(t)\}(\omega) = P(j\omega)Y(\omega). \tag{6.34}$$

Now take Fourier transforms on each side of the differential equation (6.24) and use (6.33), (6.34), to obtain

$$Q(j\omega)Y(\omega) = P(j\omega)X(\omega),$$

and therefore

$$Y(\omega) = \frac{P(j\omega)}{Q(j\omega)}X(\omega). \tag{6.35}$$

To determine the response $y(t)$ we now

(i) use the Heaviside theorem to take a partial fraction expansion in $j\omega$ of the right-hand side of (6.35);

(ii) determine the inverse Fourier transform of each partial fraction term that results from (i) using tables of Fourier transforms.

6.5 Frequency Response Function

In Section 3.8 we defined the frequency response of the system

$$\left\{\begin{array}{l} Q(D)y(t) = P(D)x(t), \\ y(0-) = \alpha_0, \quad y^{(1)}(0-) = \alpha_1, \ldots, \quad y^{(n-1)}(0-) = \alpha_{n-1}, \end{array}\right. \tag{6.36}$$

to be the function of ω given by $H(j\omega)$, where $H(s)$, the **transfer function** of the system, is in turn is given by (see Section 3.7)

$$H(s) = \frac{P(s)}{Q(s)}.$$

Thus:

(6.37)
$$H(j\omega) \triangleq \frac{P(j\omega)}{Q(j\omega)}.$$

The usefulness of the notion of frequency response was seen in Theorem 3.8.1.

In the present section we shall define the frequency response function in a more general context than that of the system (6.36), namely for a general LTI system with input signal $x(t)$ and output signal $y(t)$.

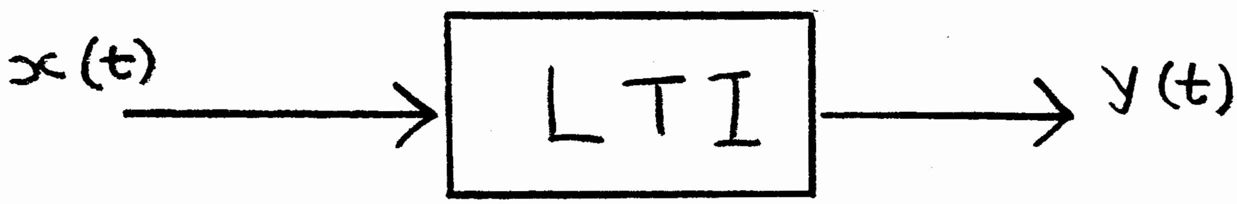


Fig. 6.8

Recall from Theorem 2.2.8 that the input and output are related by

(6.38)
$$y(t) = (h * x)(t),$$

where $h(t)$ is the impulse response of the system. We **define** the frequency response function of the LTI system with input $x(t)$ and output $y(t)$ to be the function of ω given by

(6.39)
$$H(j\omega) \triangleq \frac{\mathcal{F}\{y(t)\}(\omega)}{\mathcal{F}\{x(t)\}(\omega)}.$$

From (6.38) and convolution Property (IX) of Fourier transforms we know that

$$\mathcal{F}\{y(t)\}(\omega) = \mathcal{F}\{h(t)\}(\omega)\mathcal{F}\{x(t)\}(\omega),$$

hence it follows from (6.39) that

(6.40)
$$H(j\omega) = \mathcal{F}\{h(t)\}(\omega).$$

Thus, the frequency response function of an LTI system is also identical to the Fourier transform of its impulse response function. The crucial significance of the frequency response function $H(j\omega)$ is as follows: Suppose that the input signal for the LTI system is the complex sinusoid

$$x(t) = e^{j\omega_0 t}, \tag{6.41}$$

at some frequency ω_0. Then, from the table of Fourier transforms:

$$\mathcal{F}\{x(t)\}(\omega) = 2\pi\delta(\omega - \omega_0),$$

and, from (6.39), the corresponding output $y(t)$ has the Fourier transform:

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{y(t)\}(\omega) \\ &= H(j\omega)\mathcal{F}\{x(t)\}(\omega) \\ &= 2\pi H(j\omega)\delta(\omega - \omega_0). \end{aligned}$$

From Theorem 6.1.1 the response of the LTI system to the complex sinusoid (6.41) is therefore the signal

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)H(j\omega)e^{j\omega t}\,dt \\ &= H(j\omega_0)e^{j\omega_0 t}, \end{aligned}$$

that is,

$$y(t) = |H(j\omega_0)|e^{j[\omega_0 t + \angle H(j\omega_0)]}. \tag{6.42}$$

We see that the response of the LTI system with frequency response function $H(j\omega)$ to the complex-valued sinusoidal input (6.41) is the complex-valued sinusoidal signal $y(t)$ given by (6.42). Thus the relation between the input sinusoid and the output signal is as follows:

the output signal results from amplifying the input sinusoid by $|H(j\omega)|$ and shifting its phase by the angle $\angle H(j\omega)$.

Finally, let us look at the frequency response function for the special system (6.24). From (6.35) and (6.39) we see at once that

$$H(j\omega) = \frac{P(j\omega)}{Q(j\omega)}.$$

This is exactly consistent with our earlier definition (6.37) of frequency response for the system (6.36).

6.6 Distortionless Systems

An system is called **distortionless** when there are real constants $t_0 \geq 0$ and C such that the response $y(t)$ to an input signal $x(t)$ is of the form

$$y(t) = Cx(t - t_0). \tag{6.43}$$

That is, the output of a distortionless system must be a "replica" of the input signal, possibly scaled by a factor C and delayed in time by t_0. Typically, we want a communication system such as a microwave link, in which the input signal is the transmitted message and the output signal is the received signal, to be a distortionless system, or at least "close" to a distortionless system.

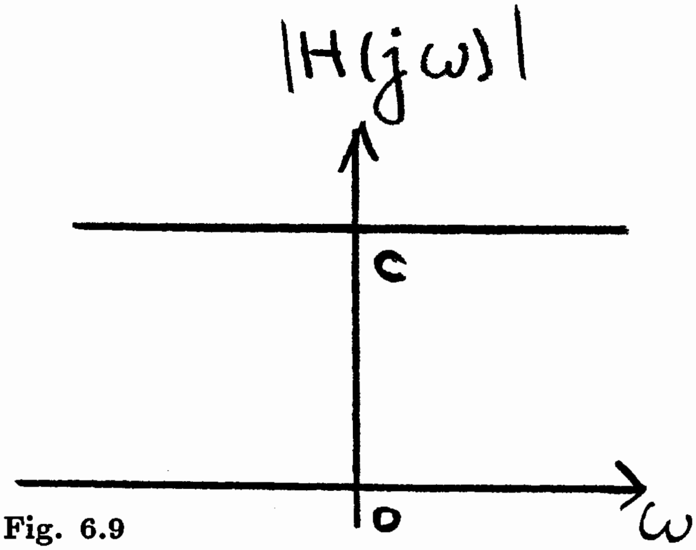
Let us determine the frequency response function of the distortionless system (6.43). From Property (III) of Fourier transforms we have

$$\begin{aligned}
\mathcal{F}\{y(t)\}(\omega) &= \mathcal{F}\{Cx(t - t_0)\}(\omega) \\
&= C\mathcal{F}\{x(t - t_0)\}(\omega) \\
&= Ce^{j\omega_0 t}\mathcal{F}\{x(t)\}(\omega),
\end{aligned}$$

thus the frequency response function is

$$\begin{aligned}
H(j\omega) &= \frac{\mathcal{F}\{y(t)\}(\omega)}{\mathcal{F}\{x(t)\}(\omega)} \\
&= Ce^{-j\omega t_0}.
\end{aligned}$$

See Fig. 6.9.



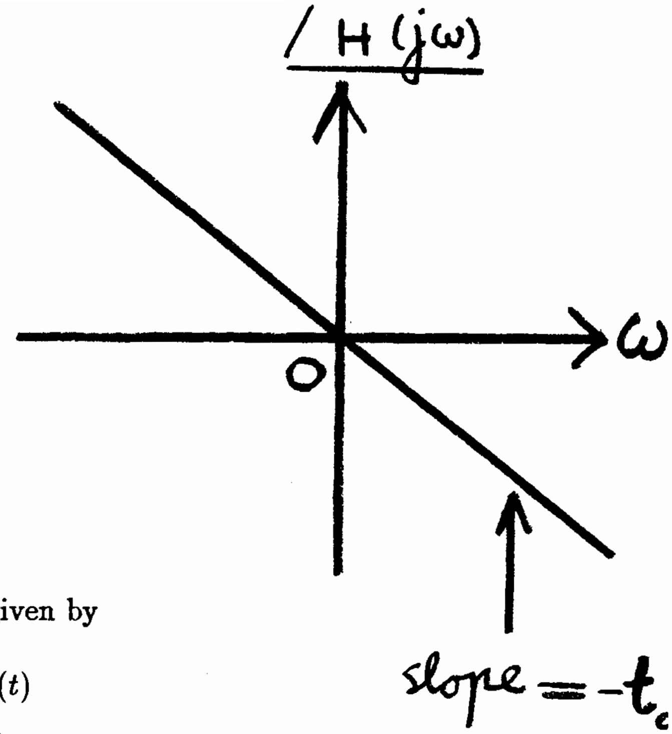


Fig. 6.9

Clearly the impulse response of a distortionless system is given by

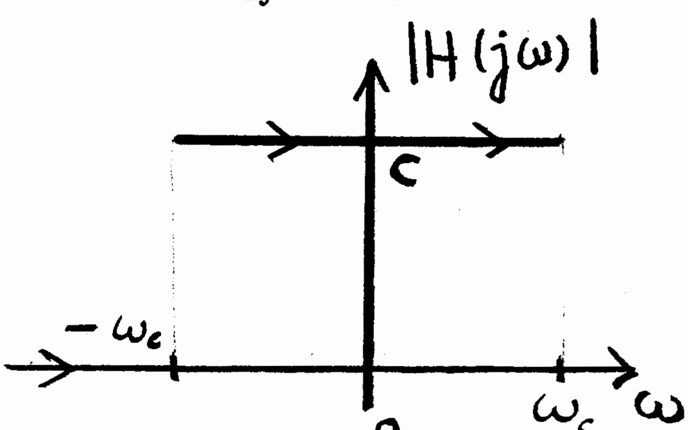
$$\begin{aligned} h(t) &= \mathcal{F}^{-1}\{H(j\omega)\}(t) \\ &= C\mathcal{F}^{-1}\left\{e^{-j\omega t_0}\right\}(t) \\ &= C\delta(t-t_0). \end{aligned}$$

6.7 The ideal Filter

An LTI system is called an **ideal low-pass filter** when its frequency response function is given by

$$H(j\omega) = \begin{cases} Ce^{-j\omega t_0}, & \text{for all } |\omega| \le \omega_C, \\ 0, & \text{for all } |\omega| > \omega_C. \end{cases} \tag{6.44}$$

Here $C > 0$, $\omega_C > 0$ and $t_0 \ge 0$ are constants called, respectively, the **gain**, the **cut-off frequency** and the **delay** of the filter.



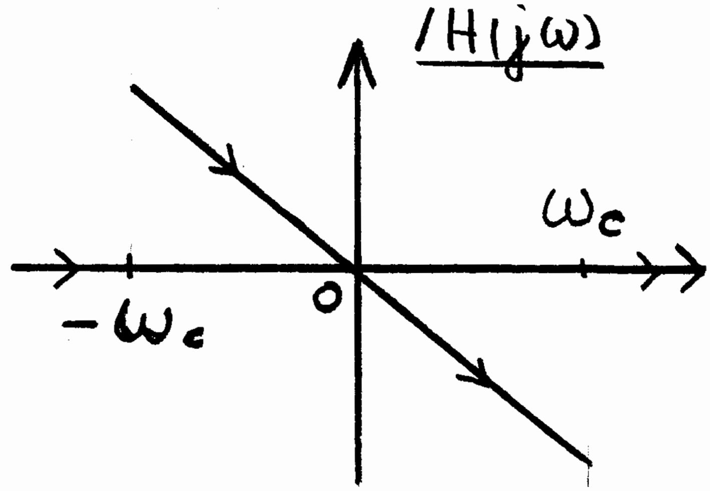


Fig. 6.10

Recalling Theorem 6.1.1, which shows us how a signal $x(t)$ is decomposed into a sum of component sinusoids

$$e^{j\omega t}$$

for frequencies over the range $-\infty < \omega < \infty$, we see that the ideal low-pass filter *removes completely* from the output signal all sinusoids for which $|\omega| > \omega_C$, leaving only those sinusoids for which $|\omega| \leq \omega_C$. We now proceed to determine the impulse response of the ideal low-pass filter. Since the frequency response function (6.44) is the Fourier transform of the impulse response function, we see from Theorem 6.1.1 that

$$
\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_C}^{\omega_C} Ce^{-j\omega t_0}e^{j\omega t}d\omega \\
&= \frac{C}{2\pi}\int_{-\omega_C}^{\omega_C} e^{j\omega(t-t_0)}d\omega \\
&= \frac{C}{2\pi}\frac{1}{j(t-t_0)}\, e^{j\omega(t-t_0)}\Big|_{-\omega_C}^{\omega_C} \\
&= \frac{C}{2\pi}\frac{1}{j(t-t_0)}\left[e^{j\omega_C(t-t_0)} - e^{-j\omega_C(t-t_0)}\right] \\
&= \left(\frac{C\omega_C}{\pi}\right)\frac{\sin[\omega_C(t-t_0)]}{\omega_C(t-t_0)}.
\end{aligned}
\tag{6.45}
$$

The form of the impulse response $h(t)$ is illustrated in Fig. 6.11. Notice that the impulse response includes a non-zero *precursor transient* for instants $t < 0$, showing that the ideal low-pass filter cannot be a causal system (recall Section 2.2.1). From a physical viewpoint we can never **make** a device for real-time filtering with frequency response function given exactly by (6.44) (equivalently, with impulse response function given by (6.45)), since such a device would have to be "clairvoyant" in the sense of giving the precursor transient in the impulse response before the impulse function even "struck" at the instants $t = 0$!

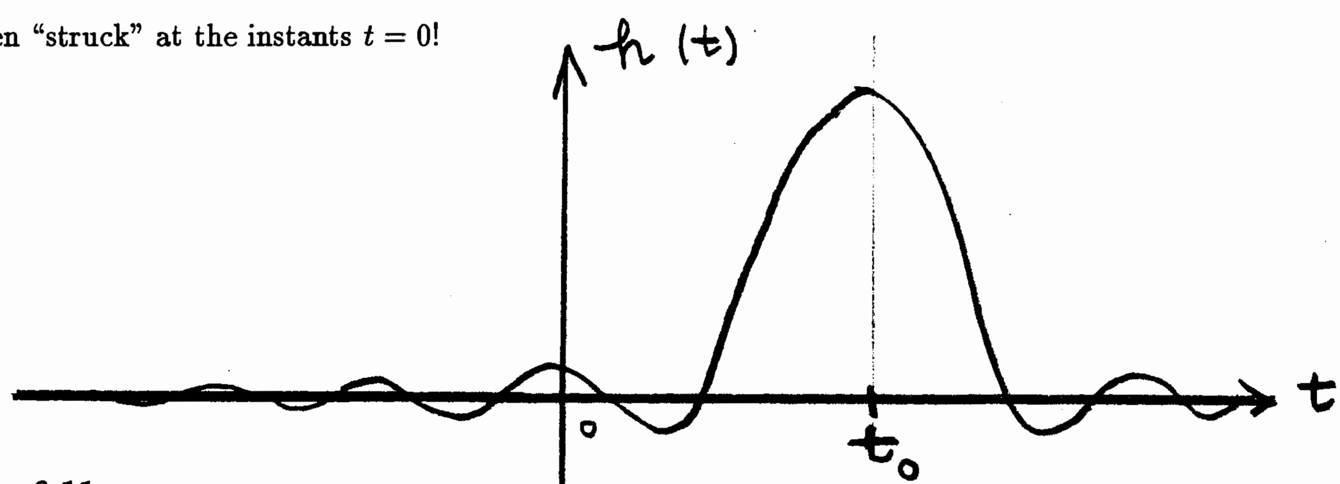


Fig. 6.11

A well-developed branch of electrical engineering called **filter design** is concerned with the problem of designing **physically realizable** LTI systems whose frequency response functions are a good approximation to that given by (6.44). Usually these LTI systems are given by the differential equation relationship

$$Q(D)y(t) = P(D)x(t),$$

and such systems can always be realized (i.e. made) from commonly available components such as resistors, capacitors, inductors and operational amplifiers. The task of filter design then becomes one of appropriately choosing or "designing" the orders and the coefficients of the polynomials $P(D)$ and $Q(D)$ so that the resulting LTI system is, in some sense, a good approximation of the ideal low-pass filter.

6.8 Application: Amplitude Modulation

Many signals $x(t)$ which occur naturally in electrical engineering are **band-limited** in the following sense: there is some angular frequency ω_B such that

$$X(\omega) = 0, \qquad \text{for all } |\omega| \geq \omega_B,$$

where $X(\omega)$ is the Fourier transform of $x(t)$. Physically, this means that the signal $x(t)$ does not include any component sinusoids of an angular frequency greater than ω_B.

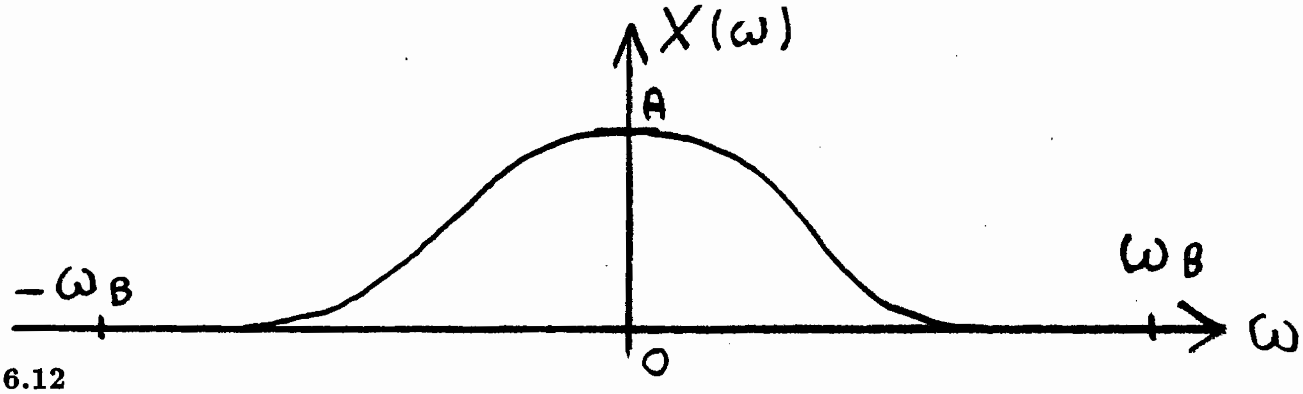


Fig. 6.12

For example, audio signals are typically band-limited by about $\omega_B = 25 \times 10^3$ rads/sec. Suppose we want to transmit a band-limited signal $x(t)$ through free space from a transmission station to a receiver, as in radio and television transmission. In so-called *double-side-band-suppressed-carrier* AM transmission the signal $x(t)$ is the input to a **modulator** whose corresponding output is the signal

$$z(t) = x(t)\cos(\omega_0 t),$$

where ω_0 is some (typically very large) angular frequency called the **carrier angular frequency**. In particular, ω_0 is much greater that ω_B. See Fig. 6.13 and Fig. 6.14.

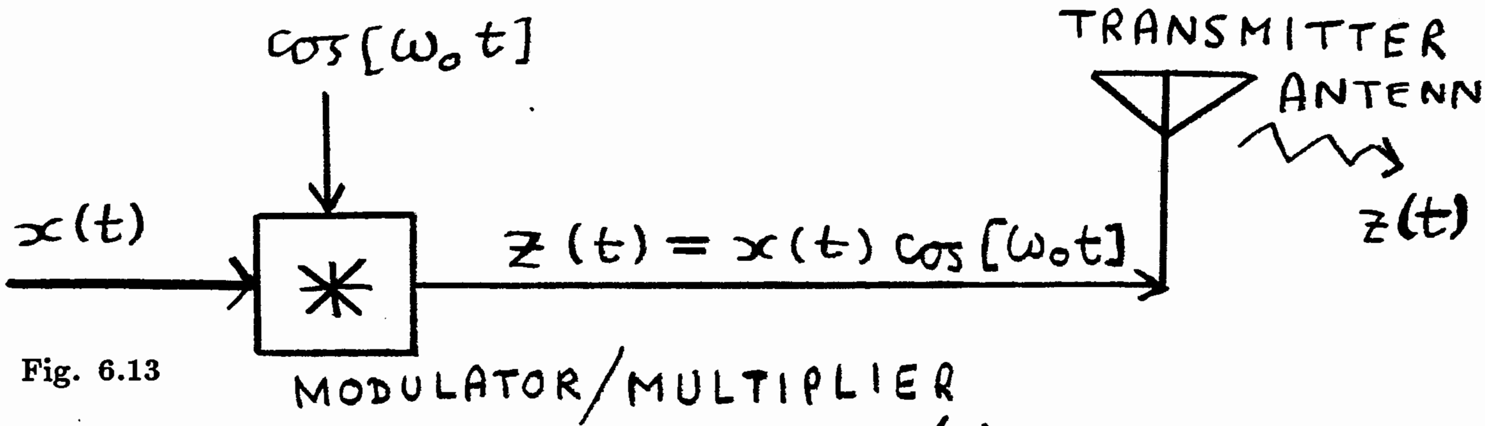


Fig. 6.13

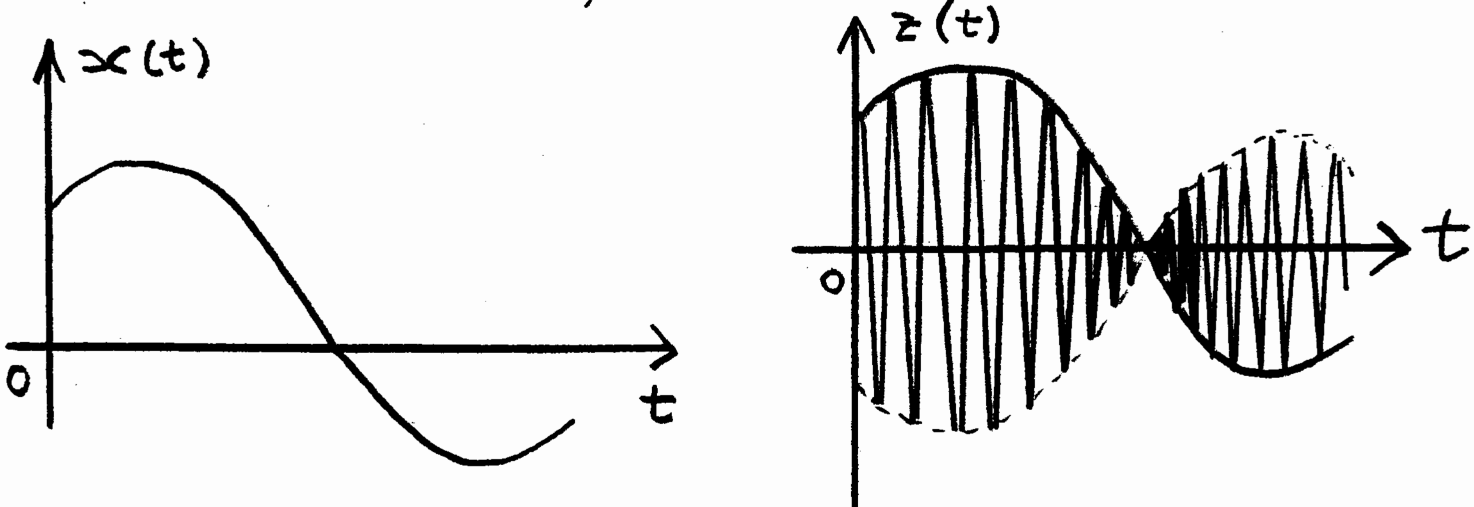


Fig. 6.14

Notice that a modulator is clearly a linear, although not time-invariant, system. The reason for introducing modulation is that the high-frequency signal $z(t)$ travels much more efficiently through free space than would the lower-frequency signal $x(t)$. In fact, we shall suppose that the transmitted signal $z(t)$ is also the signal that appears at the antenna of the receiver, so that we neglect distortion, attenuation, and the addition of noise to the signal as it passes through free space from the transmitter to the receiver. The problem is then to extract the original signal $x(t)$ from the signal $z(t)$ that appears at the antenna of the receiver. This process of extraction is called **demodulation**, and Fourier transforms are an essential tool for understanding how demodulation works. The first step in the demodulation process is to multiply the received signal $z(t)$ by the sinusoid $\cos(\omega_0 t)$ to get the signal

$$m(t) \triangleq z(t)\cos(\omega_0 t) \tag{6.46}$$

$$
\begin{aligned}
&= x(t)\cos^2(\omega_0 t) \\
&= \frac{x(t)}{2}[1+\cos(2\omega_0 t)] \\
&= \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(2\omega_0 t).
\end{aligned}
$$

(In a radio or television receiver there is a "local oscillator" which generates a replica $\cos(\omega_0 t)$ of the carrier wave-form, and then a diode circuit is used to accomplish the multiplication of the received signal $z(t)$ by this wave-form - see Fig. 6.15.).

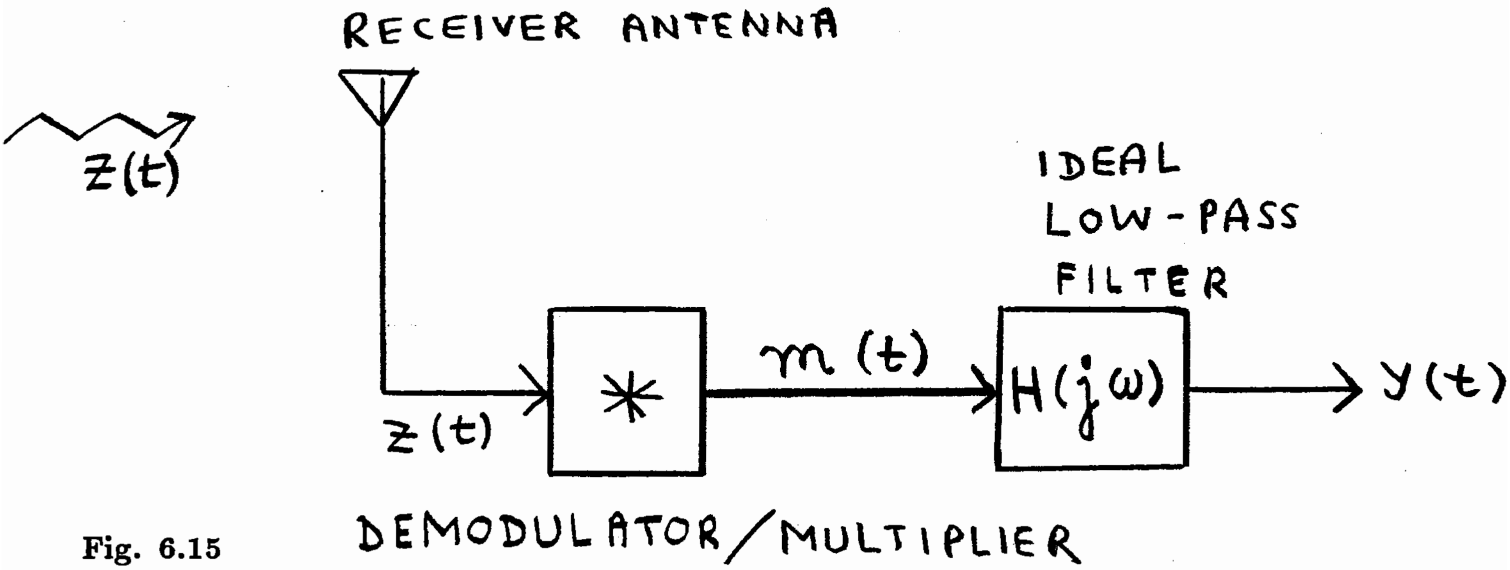


Fig. 6.15

Using the frequency-shift Property (IV) of Fourier transforms to find the Fourier transform of the signal $m(t)$ at the output of the multiplier we get

$$
\begin{aligned}
&(6.47) \qquad M(\omega) \triangleq \mathcal{F}\{m(t)\}(\omega) \\
&(6.48) \qquad = \mathcal{F}\left\{\frac{1}{2}x(t)\right\}(\omega) + \mathcal{F}\left\{\frac{1}{2}x(t)\cos(2\omega_0 t)\right\}(\omega) \\
&\qquad\qquad = \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega - 2\omega_0) + \frac{1}{4}X(\omega + 2\omega_0).
\end{aligned}
$$

Thus, if the frequency spectrum of $x(t)$ is as shown in Fig. 6.12 then the frequency spectrum of the signal $m(t)$ at the output of the multiplier is necessarily of the form shown in Fig. 6.16.

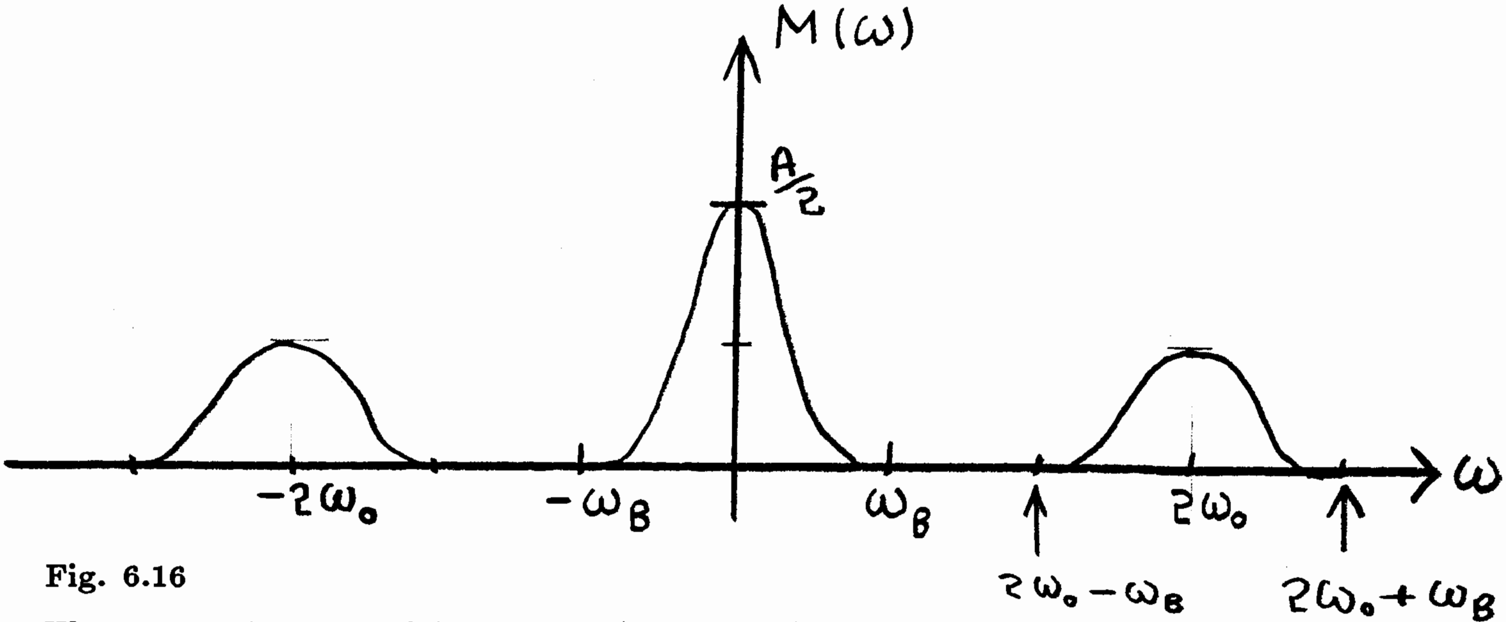


Fig. 6.16

We next pass the output of the multiplier (see Fig. 6.15) though an ideal low-pass filter of the form (6.44) with $C = 1$ and $t_0 = 0$, that is

$$H(j\omega) = \begin{cases} 1, & \text{for all } |\omega| \leq \omega_B, \\ 0, & \text{otherwise.} \end{cases}$$

See Fig. 6.17.

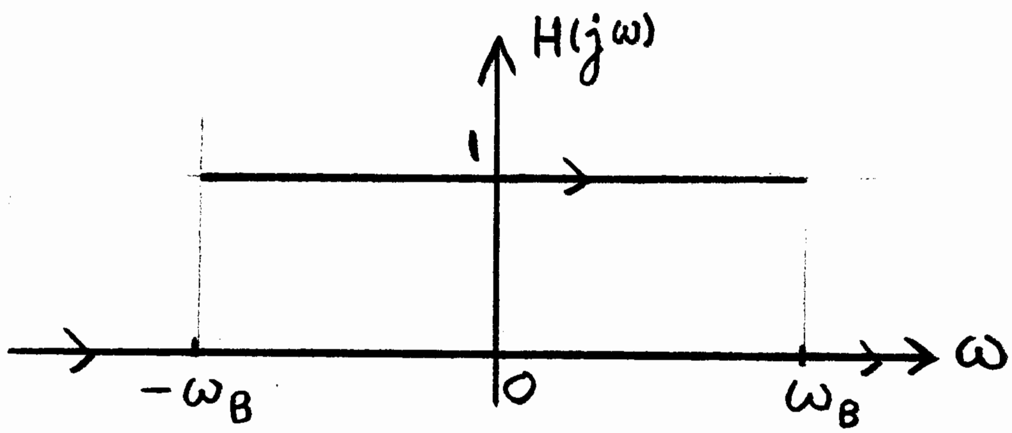


Fig. 6.17

If $y(t)$ denotes the output signal of the ideal low-pass filter then, from (6.47),

$$\begin{aligned} Y(\omega) &= H(j\omega)M(\omega) \\ &= \frac{1}{2}H(j\omega)X(\omega) + \frac{1}{4}H(j\omega)X(\omega - 2\omega_0) + \frac{1}{4}H(j\omega)X(\omega + 2\omega_0). \end{aligned} \tag{6.49}$$

Since $X(\omega)$ is bandlimited by ω_B (see Fig. 6.12) and the low-pass filter $H(j\omega)$ has cut-off frequency ω_B, we see that

$$H(j\omega)X(\omega) = X(\omega), \qquad \text{for all } \omega. \tag{6.50}$$

See Fig. 6.18.

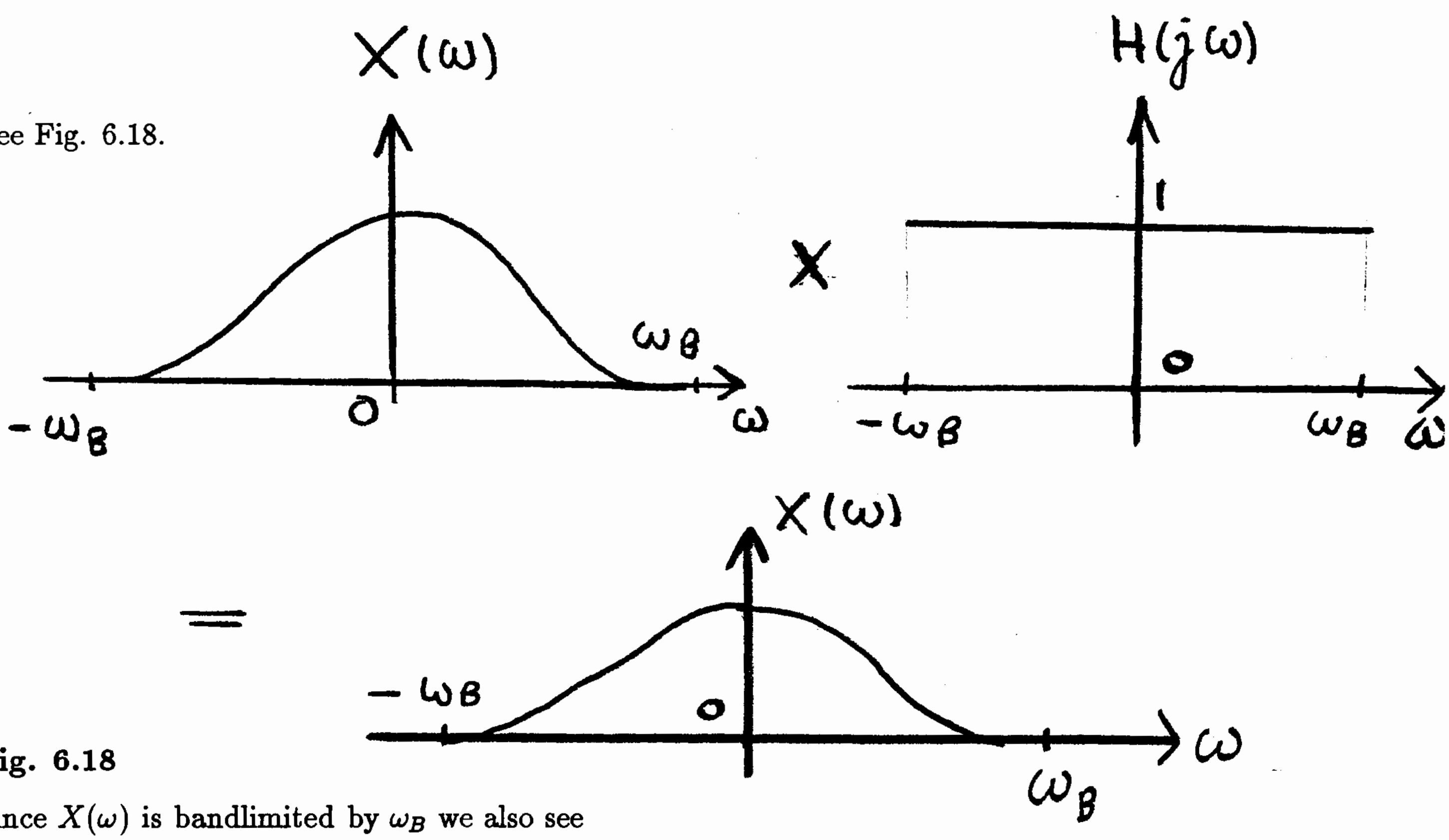


Fig. 6.18

Since $X(\omega)$ is bandlimited by ω_B we also see

$$X(\omega - 2\omega_0) = 0, \qquad \text{for all } \omega \leq 2\omega_0 - \omega_B \text{ or } \omega \geq 2\omega_0 + \omega_B.$$

Now the carrier frequency ω_0 is chosen to be much greater than ω_B so that certainly we have

$$2\omega_0 - \omega_B > \omega_B.$$

Since $H(j\omega) = 0$ for all $\omega \geq \omega_B$ it follows that

(6.51) $$H(j\omega)X(\omega - 2\omega_0) = 0, \qquad \text{for all } \omega.$$

See Fig. 6.19.

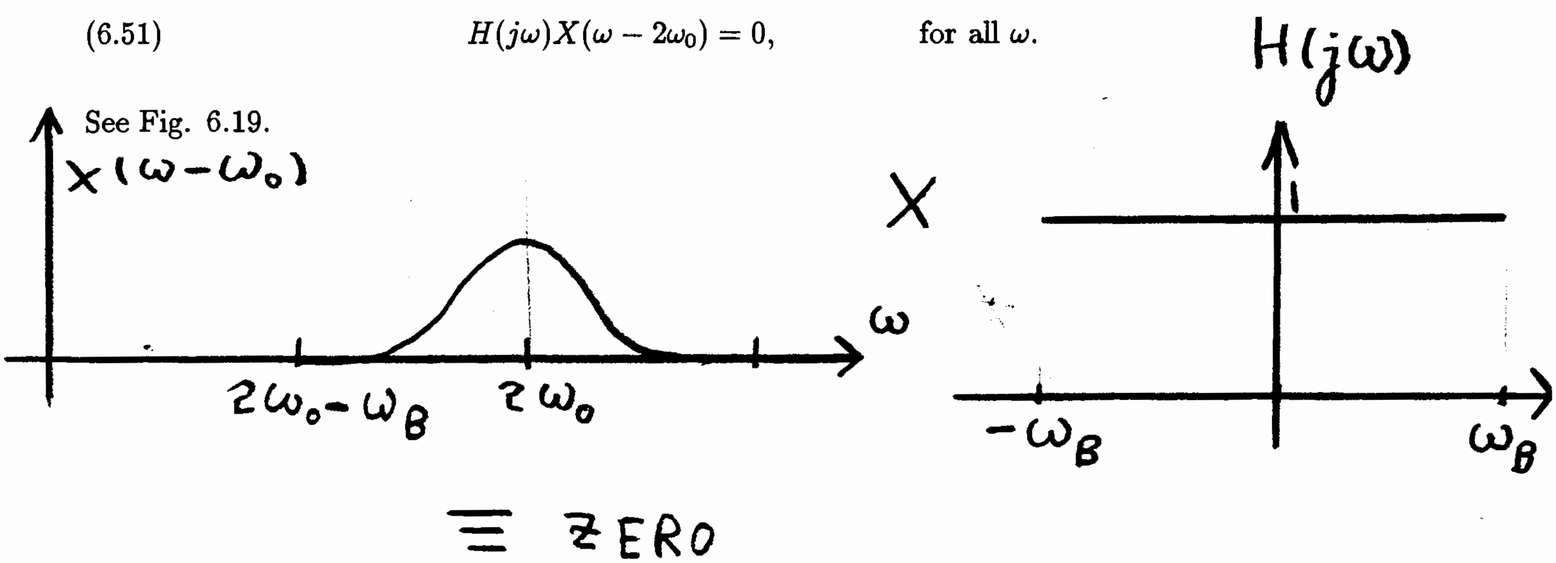


Fig. 6.19

In exactly the same way we get

$$H(j\omega)X(\omega + 2\omega_0) = 0, \qquad \text{for all } \omega. \tag{6.52}$$

Combining (6.49), (6.50), (6.51), (6.52) we find that

$$Y(\omega) = \frac{1}{2}X(\omega), \qquad \text{for all } \omega.$$

Thus, the output signal $y(t)$ of the low-pass filter in Fig. 6.14 is an exact replica of the original signal $x(t)$, just scaled by a factor of $1/2$.