UNIVERSITY OF WATERLOO
ECE 784 - STAT 902 Midterm Examination Winter 2011
Instructor: Andrew Heunis
2.30p.m. - 3.45 p.m. Thursday, 10 March, 2011

Answer all five questions. Point allocation for each question is shown. Total marks = 70.
Aids permitted: printed class notes only. Duration of examination: 1 hour 15 minutes

1. [10] X is a random variable on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-algebra. X takes values in $(0, \infty)$, and is such that $E[X] < \infty$ and $E[X^{-\alpha}] < \infty$ for some constant $\alpha \in [0, \infty)$. Establish that

$$\frac{1}{(E[X \mid \mathcal{G}])^\alpha} \leq E\left[\frac{1}{X^\alpha} \,\middle|\, \mathcal{G}\right].$$

2. [15] $\{X_n, \quad n = 1, 2, \ldots\}$ is a sequence of nonnegative and uniformly bounded random variables on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ is a sub-σ-algebra (i.e. $0 \leq X_n(\omega) \leq c$ for all $\omega \in \Omega$ and $n = 1, 2, \ldots$, for some constant $c \in [0, \infty)$). Establish

$$\limsup_{n \to \infty} E[X_n \mid \mathcal{G}] \leq E\left[\limsup_{n \to \infty} X_n \,\middle|\, \mathcal{G}\right].$$

3. [15] $\{\mathcal{F}_n, \quad n = 0, 1, 2, \ldots\}$ is a filtration in the probability space $(\Omega, \mathcal{F}, P)$, and Q is a probability measure Q on $(\Omega, \mathcal{F})$ such that for each $n = 0, 1, 2, \ldots$ one has

$$P(A) = Q(A), \quad \text{for all } A \in \mathcal{F}_n.$$

Establish that $P(A) = Q(A)$ for all $A \in \mathcal{F}_\infty$.
Hint: Use Theorem 1.5.4 of the notes.

4. [10] The processes $\{(X_n, \mathcal{F}_n); n = 0, 1, 2, \ldots, +\infty\}$ and $\{(Y_n, \mathcal{F}_n); n = 0, 1, 2, \ldots, +\infty\}$ are **closed** square-integrable martingales on the probability space $(\Omega, \mathcal{F}, P)$ (see Definition 2.3.4 of the notes), and T is a $\{\mathcal{F}_n\}$-stopping time T. Show that

$$E[X_T Y_\infty] = E[X_T Y_T] = E[X_\infty Y_T].$$

Hint: Use Theorem 2.3.9 of the notes.

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5. [20] $\{X_n,\ n = 1, 2, \ldots\}$ is a sequence of independent random variables on $(\Omega, \mathcal{F}, P)$. Put

$$\mathcal{G}_k = \sigma\{X_k, X_{k+1}, X_{k+2}, \ldots\} \text{ for all } k = 1, 2 \ldots, \quad \text{and } \mathcal{G} \triangleq \bigcap_{1 \le k < \infty} \mathcal{G}_k,$$

$$\mathcal{F}_n = \sigma\{X_1, X_2, \ldots, X_n\}, \text{ for all } n = 1, 2, \ldots, \quad \mathcal{F}_\infty = \sigma\left\{\cup_{1 \le n < \infty} \mathcal{F}_n\right\}.$$

($\mathcal{G}$ is called the "tail σ-algebra" of the sequence $\{X_n\}$). Establish the following:

(a) $\mathcal{G}_k \subset \mathcal{F}_\infty$, $k = 1, 2, \ldots$

(b) $\mathcal{F}_n$ and $\mathcal{G}$ are P-independent for all $n = 1, 2, \ldots$

(c) if Z is integrable and $\mathcal{G}$-measurable then $Z = E[Z]$ a.s.

Hint: Use the Levy martingale convergence theorem (Theorem 2.7.6 of the notes) for (c).

1. Given $X : (\Omega, \mathcal{F}, P) \longrightarrow (0, \infty)$

s.t.

$$E[X] < \infty \quad \text{———} \; (1)$$

$$E\left[\frac{1}{X^\alpha}\right] < \infty \quad \text{for some } \alpha \in [0, \infty) \; \text{——} \; (2)$$

Define $c : (0, \infty) \longrightarrow (0, \infty)$ by

$$c(x) = \frac{1}{x^\alpha}, \quad x \in (0, \infty) \; \text{——} \; (3)$$

Then $c(\cdot)$ is clearly convex on $I = (0, \infty)$ and

$$E\left[|c(X)|\right] \overset{(3)}{=} E\left[\frac{1}{X^\alpha}\right] < \infty \; \text{——} \; (4)$$

From (1) (4) and Thm 1.4.20

$$c\left(E[X \mid \mathcal{G}]\right) \leq E\left[c(X) \mid \mathcal{G}\right] \; \text{——} \; (5)$$

From 5 3

$$\frac{1}{\left(E[X\mid\mathcal{G}]\right)^{\alpha}} \leq E\left[\frac{1}{X^{\alpha}} \,\middle|\, \mathcal{G}\right] \quad a.s.$$

2. Given

$$0 \leq X_n(\omega) \leq c < \infty, \quad \text{all } \omega \in \Omega, \quad \text{all } n = 1, 2, 3 \quad \text{—— (1)}$$

Then

$$c \geq c - X_n \geq 0 \quad \text{—— (2)}$$

From (2) and Thm 1.4.19 (b) [Fatou for conditional expectation]

$$\liminf_{n \to \infty} E[c - X_n \mid \mathcal{G}] \geq E[\overbrace{\liminf_{n \to \infty} (c - X_n)}^{\leq c < \infty \text{ from (2)}} \mid \mathcal{G}] \quad \text{—— (3)}$$

For any sequence $\{x_n\}$ of real numbers

$$\liminf_{n \to \infty} (c - x_n) = c - \limsup_{n \to \infty} x_n \quad \text{—— (4)}$$

$$\liminf_{n\to\infty} E[c - X_n \mid \mathcal{G}]$$

$$= \liminf_{n\to\infty} \Big(\underbrace{E[c \mid \mathcal{G}]}_{= c} - E[X_n \mid \mathcal{G}] \Big)$$

$$\stackrel{(4)}{=} c - \limsup_{n\to\infty} E[X_n \mid \mathcal{G}] \quad \text{—— (5)}$$

and

$$E\Big[\liminf_{n\to\infty} (c - X_n) \,\Big|\, \mathcal{G} \Big]$$

$$\stackrel{(4)}{=} E\Big[c - \limsup_{n\to\infty} X_n \,\Big|\, \mathcal{G} \Big]$$

$$= E[c \mid \mathcal{G}] - E\Big[\limsup_{n\to\infty} X_n \,\Big|\, \mathcal{G} \Big]$$

$$= c - E\Big[\limsup_{n\to\infty} X_n \,\Big|\, \mathcal{G} \Big] \quad \text{—— (6)}$$

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$$c - \limsup_{n\to\infty} E[X_n \mid \mathcal{G}] \geq c - E[\limsup_n X_n \mid \mathcal{G}]$$

ie

$$\limsup_{n\to\infty} E[X_n \mid \mathcal{G}] \leq E[\limsup_n X_n \mid \mathcal{G}].$$

3.

$\{\mathcal{F}_n,\ n = 0,1,2 \ldots\}$ is a filtration in $(\Omega, \mathcal{F}, P)$ ___ (1)

For all $n = 1,2 \ldots$ have

$$P(A) = Q(A), \quad \text{all } A \in \mathcal{F}_n \text{ ___ } (2)$$

Put

$$\mathcal{D} \triangleq \{A \in \mathcal{F} : A \in \mathcal{F}_n \text{ for some } n = 0,1,2 \ldots\}$$

$$\equiv \bigcup_{0 \le n < \infty} \mathcal{F}_n \text{ ___ } (3)$$

Then

$$\mathcal{F}_\infty \triangleq \sigma\{\mathcal{D}\} \text{ ___ } (4)$$

Now show

$$\mathcal{D} \text{ is a } \pi\text{-class ___ } (5)$$

Fix A_1 & $A_2 \in \mathcal{D}$: from (3) have

$$A_1 \in \mathcal{F}_{n_1} \quad \& \quad A_2 \in \mathcal{F}_{n_2}$$

for integers $n_1 \,\&\, n_2 \in \{0,1,2 \ldots\}$ —— (6)

thus

$$A_1 \in \mathcal{F}_{n_3} \quad \& \quad A_2 \in \mathcal{F}_{n_3} \quad \text{for } n_3 \doteq \max(n_1, n_2)$$ —— (7)

(since the $\mathcal{F}_n$ are <u>increasing</u> with n - see (1))

Then

$$A_1 \cap A_2 \overset{(7)}{\in} \mathcal{F}_{n_3} \overset{(3)}{\subset} \mathcal{D}$$

ie $\quad A_1 \cap A_2 \in \mathcal{D}$

which gives (5).

Put

$$\mathcal{C} \doteq \{A \in \mathcal{F}_\infty : P(A) = Q(A)\}$$ —— (8)

Then

$$A \in \mathcal{D} \overset{(3)}{\Longrightarrow} A \in \mathcal{F}_n \text{ for some } n \in \{0,1,2 \ldots\}$$

$$\Rightarrow \quad P(A) = Q(A)$$

i.e. $\quad P(A) = Q(A) \quad$ all $\quad A \in \mathcal{D}$ ___ (9)

ie $\quad \mathcal{D} \overset{(9)(8)}{\subset} \mathcal{C}$ ___ (10)

Next show

$\mathcal{C}$ is a λ-class over Ω ___ (11)

Have $\quad Q(\Omega) = 1 = P(\Omega)$ ___ (12)

(P & Q are given <u>probability</u> measures)

ie $\quad \Omega \overset{(12)}{\in} \mathcal{C}$ ___ (13)

Fix $\quad A$ & $B \in \mathcal{C}$ with $A \subset B$ ___ (14)

Then $\quad B = A \cup (B - A)$ & $A \cap (B - A) = \emptyset$ ___ (15)

3.4

ie $$\underline{P}(B) \overset{(15)}{=} \underline{P}(A) + \underline{P}(B - A)$$

ie $$\underline{P}(B - A) = \underline{P}(B) - \underline{P}(A) \quad \text{—— (16)}$$

similarly
$$Q(B - A) = Q(B) - Q(A) \quad \text{—— (17)}$$

From (14) (8)

$$\underline{P}(A) = Q(A) \qquad \underline{P}(B) = Q(B) \quad \text{—— (18)}$$

ie
$$\underline{P}(B - A) \overset{(16)}{=} \underline{P}(B) - \underline{P}(A) \overset{(18)}{=} Q(B) - Q(A) \overset{(17)}{=} Q(B - A) \quad \text{—— (19)}$$

From (19) $B - A \in \mathcal{C}$ when A & $B \in \mathcal{C}$

with $A \subset B$ —— (20)

Now fix

$$A_n \in \mathcal{C} \quad \text{s.t.} \quad A_n \subset A_{n+1} \quad \& \text{ put } \quad A \triangleq \bigcup_{1 \le n < \infty} A_n \quad \text{—— (21)}$$

Since $\underline{P}$ & Q are <u>countably</u> additive on $\mathcal{F}$, see from (21) that

$$\left.\begin{aligned} \underline{P}(A) &= \lim_n \underline{P}(A_n) \quad \text{—— (22)} \\ Q(A) &= \lim_n Q(A_n) \quad \text{—— (23)} \end{aligned}\right\} \text{ see thm. 1.2.5.}$$

Also

$$\underline{P}(A_n) = Q(A_n) \quad \text{—— (24)}$$

(from (8) with $A_n \in \mathcal{C}$ - see (21))

From (22) (23) and (24):

$$Q(A) = \underline{P}(A) \quad \text{—— (25)}$$

From (21) and (25) have shown:

if $A_n \in \mathcal{C}$ with $A_n \subset A_{n+1}$

$$\text{then} \quad \bigcup_{1 \le n < \infty} A_n \in \mathcal{C} \quad \text{—— (26)}$$

From (26) (20) (13) get (11) (see Definition 1.5.2).

From (11) (10) (5) and Thm 1.5.4:

$$\sigma\{\mathcal{D}\} \subset \mathcal{C} \quad \text{——} \quad (27)$$

ie

$$\mathcal{F}_\infty \overset{(4)}{=} \sigma\{\mathcal{D}\} \overset{(27)}{\subset} \mathcal{C} \overset{(8)}{\subset} \mathcal{F}_\infty$$

ie

$$\mathcal{F}_\infty = \mathcal{C}. \quad \text{——} \quad (28)$$

Result follows from (28) and (8).

4. Fix the $\{\mathcal{F}_n\}$-stopping time T.

Since
$$\{(X_n, \mathcal{F}_n),\ n = 0, 1, 2, \dots, +\infty\}$$
$$\{(Y_n, \mathcal{F}_n),\ n = 0, 1, 2, \dots, +\infty\}$$
are closed martingales, Theorem 2.3.9 gives
$$X_T = E[X_\infty \mid \mathcal{F}_T] \qquad Y_T = E[Y_\infty \mid \mathcal{F}_T] \text{ — } (1)$$

Also by Jensen for conditional expectation
$$|X_T|^2 \overset{(1)}{=} \left(E[X_\infty \mid \mathcal{F}_T]\right)^2 \le E[|X_\infty|^2 \mid \mathcal{F}_T] \text{ — } (2)$$

since postulated square-integrability gives
$$E[|X_\infty|^2] < \infty \text{ — } (3)$$

Then
$$E[|X_T|^2] \overset{(2)}{\le} E\left[E[|X_\infty|^2 \mid \mathcal{F}_T]\right] = E[|X_\infty|^2] < \infty$$
(4)

Similarly

$$E\left[|Y_T|^2\right] < \infty \quad \text{—— } (5)$$

Then

$$E\left[|X_\infty Y_T|\right] \underset{\text{cauchy-schwarz}}{\leq} \sqrt{E\left[|X_\infty|^2\right]}\sqrt{E\left[|X_T|^2\right]} \overset{(3)(4)}{<} \infty \quad \text{—— } (6)$$

Similarly

$$E\left[|X_T Y_\infty|\right] < \infty \quad \text{—— } (7)$$

and

$$E\left[|X_T Y_T|\right] < \infty \quad \text{—— } (8)$$

From (6) (7) (8)

$E[X_\infty Y_T]$, $E[X_T Y_\infty]$ and $E[X_T Y_T]$ are defined

and exist in $\mathbb{R}$ —— (9)

Then

$$E[X_T Y_\infty] \overset{(i)}{=} E\left[E(X_\infty | \mathcal{F}_T) Y_\infty\right]$$

$$= E\left\{ E\left[\underbrace{E[X_\infty \mid \mathcal{F}_T]}_{\mathcal{F}_T\text{-measurable}} Y_\infty \,\Big|\, \mathcal{F}_T \right] \right\}$$

$$= E\left\{ E[X_\infty \mid \mathcal{F}_T] \cdot E[Y_\infty \mid \mathcal{F}_T] \right\}$$

$$\overset{(i)}{=} E[X_T Y_T] \quad \text{—— } (10)$$

switching roles of X and Y at (10)

$$E[X_\infty Y_T] = E[X_T Y_T] \quad \text{— } (11)$$

as required.

5. Given indep. rvs X_n, $n = 1, 2, 3 \ldots$ on $(\Omega, \mathcal{F}, P)$

$$\mathcal{G}_k \triangleq \sigma\{X_k, X_{k+1}, \ldots\} \quad \text{—— (1)}$$

$$\mathcal{G} \triangleq \bigcap_{1 \le k < \infty} \mathcal{G}_k \quad \text{—— (2)}$$

$$\mathcal{F}_n \triangleq \sigma\{X_1, X_2 \ldots X_n\}, \quad n \ge 1 \quad \text{—— (3)}$$

$$\mathcal{F}_\infty \triangleq \sigma\Big\{\bigcup_{1 \le n < \infty} \mathcal{F}_n\Big\} \quad \text{—— (4)}$$

(a) Fix some +ve integer k.

From (3) see that

$$X_k \text{ is } \mathcal{F}_n\text{-meas for all } n \ge k \quad \text{—— (5)}$$

From (5) and $\mathcal{F}_n \subset \mathcal{F}_\infty$ see that

$$X_k \text{ is } \mathcal{F}_\infty\text{-meas for all } k \ge 1 \quad \text{—— (6)}$$

From 6

$$\underbrace{\sigma\{X_K, X_{K+1} \dots\}}_{\mathcal{G}_K} \subset \mathcal{F}_\infty$$

i.e. $\mathcal{G}_K \subset \mathcal{F}_\infty, \quad K = 1,2 \dots$ ——— 7

(b) Since the sequence $\{X_K, K = 1, 2 \dots\}$ is indep. we know that

$$\underbrace{\sigma\{X_1, \dots X_n\}}_{\mathcal{F}_n} \perp\!\!\!\perp \underbrace{\sigma\{X_{n+1}, X_{n+2} \dots\}}_{\mathcal{G}_{n+1}}$$

ie

$\mathcal{F}_n$ and $\mathcal{G}_{n+1}$ are independent for all $n \geq 1$ ——— 8

Since $\mathcal{G} \subset \mathcal{G}_{n+1}$, it follows from 8 that

$\mathcal{F}_n$ and $\mathcal{G}$ are indep. for all $n \geq 1$ ——— 9

(c) Z is $\mathcal{G}$-meas s.t. $E|Z| < \infty$ ___ (10)

From Th 2.7.6

$$\lim_n E[Z | \mathcal{F}_n] = E[Z | \mathcal{F}_\infty] \quad \text{\_\_\_ (11)}$$

From (2) (7) have

$$\mathcal{G} \subset \mathcal{F}_\infty \quad \text{\_\_\_ (12)}$$

In view of (12) and (10) have

Z is $\mathcal{F}_\infty$-measurable (13)

$\therefore$

$$E[Z | \mathcal{F}_\infty] \overset{(13)}{=} Z \quad \text{\_\_\_ (14)}$$

In view of (10) and (9) also have

$\sigma\{Z\}$ and $\mathcal{F}_n$ are indep ___ (15)

Then

$$E[Z \mid \mathcal{F}_n] = E[Z] \quad \text{—— (16)}$$

see (15) and Thm 1.4.15

From (16) (14) (11)

$$E[Z] = Z \quad \text{a.s.}$$