4.8 Problems

In each of the following problems $\{\mathcal{F}_t,\ t \in [0,\infty)\}$ is a given filtration in the probability space $(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_0$ includes all P-null events in $\mathcal{F}$.

Problem 4.8.1 Suppose that $\{(X(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ is a progressively measurable $\mathbb{R}$-valued process and X_∞ is a $\mathcal{F}_\infty$-measurable random variable on $(\Omega, \mathcal{F}, P)$, such that $E|X(T)| < \infty$ and $EX(T) = 0$ for each $\{\mathcal{F}_t\}$-stopping time T. Show that $\{(X(t), \mathcal{F}_t);\ t \in [0,\infty]\}$ is a *uniformly integrable* martingale. Hint : Extend the argument used for Lemma 2.3.13.

Problem 4.8.2 Suppose that $\{(X_t, \mathcal{F}_t);\ t \in [0,\infty)\}$ is a right-continuous martingale on the probability space $(\Omega, \mathcal{F}, P)$. Show that $\{(X_t, \mathcal{F}_{t+});\ t \in [0,\infty)\}$ is also a martingale on $(\Omega, \mathcal{F}, P)$.
Hint: Use Theorem 2.7.9.

Problem 4.8.3 Suppose that $\{(X_t, \mathcal{F}_t);\ t \in [0,\infty)\}$ is a martingale on $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{G}_t,\ t \in [0,\infty)\}$ is a filtration in $(\Omega, \mathcal{F}, P)$ with $\mathcal{G}_t \subset \mathcal{F}_t$, $\forall\, t \in [0,\infty)$. Show that $\{(Y_t, \mathcal{G}_t);\ t \in [0,\infty)\}$ is a martingale, for $Y_t \triangleq E[X_t \mid \mathcal{G}_t]$, $\forall\, t \in [0,\infty)$.

Problem 4.8.4 Suppose that $X \in \mathbf{M}^c_{loc}(\{\mathcal{F}_t\}, P)$. Show that $X \in \mathbf{M}^c_{loc}(\{\mathcal{F}^X_t\}, P)$, for

$$\mathcal{F}^X_t \triangleq \sigma\{X_u,\ u \in [0,t]\}.$$

Problem 4.8.5 Suppose $\{(X(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ is a right-continuous martingale such that, for each $(t,\omega) \in [0,\infty) \otimes \Omega$, $X(t,\omega)$ takes values in the two-member set $\{a, b\}$, for constants $a < b$. Show that

$$P\{X(t) = X(0),\ \forall\, t \in [0,\infty)\} = 1.$$

that is, the process $\{X(t);\ t \in [0,\infty)\}$ is a.s. constant. Hint: Affinely transform $\{X(t)\}$ to a process $\{Y(t)\}$ which takes values in the two-member set $\{0,1\}$.

Problem 4.8.6 Suppose that $\{Z_t;\ t \in [0,\infty)\}$ is a jointly measurable $\mathbb{R}$-valued process on a complete probability space $(\Omega, \mathcal{F}, P)$ with $E|Z_t| < \infty$, $\forall\, t \in [0,\infty)$, and $\{\mathcal{G}_t,\ t \in [0,\infty)\}$ is some filtration in $(\Omega, \mathcal{F}, P)$ such that $\mathcal{G}_0$ includes all P-null events in $\mathcal{F}$. A fundamental result of Dellacherie and Meyer shows that there exists a $\{\mathcal{G}_t\}$-progressively measurable process $\{(\hat{Z}_t, \mathcal{G}_t);\ t \in [0,\infty)\}$ such that

$$\hat{Z}_t = E[Z_t \mid \mathcal{G}_t] \qquad \text{a.s.}$$

for each $t \in [0,\infty)$. This process is called an **optional projection** of the process $\{Z_t;\ t \in [0,\infty)\}$ onto the filtration $\{\mathcal{G}_t,\ t \in [0,\infty)\}$. Now do the following: Suppose that (i) $\{\mathcal{F}_t,\ t \in [0,\infty)\}$ is a filtration in $(\Omega, \mathcal{F}, P)$ such that $\mathcal{G}_t \subset \mathcal{F}_t$, $\forall\, t \in [0,\infty)$; (ii) $\{(Y_t, \mathcal{F}_t);\ t \in [0,\infty)\}$ is a progressively measurable $I\!\!R$-valued process and there is a constant $C \in [0,\infty)$ such that $|Y_t(\omega)| \le C < \infty$, $\forall\, (t,\omega) \in [0,\infty) \otimes \Omega$; (iii) $X \in \mathbf{M}(\{\mathcal{F}_t\}, P)$, where $\{X_t;\ t \in [0,\infty)\}$ is a jointly measurable and $I\!\!R$-valued process on $(\Omega, \mathcal{F}, P)$. If

$$Z_t \triangleq X_t + \int_0^t Y_s\, ds, \qquad \forall\, t \in [0,\infty),$$

and $\{\hat{Z}_t;\ t \in [0,\infty)\}$, $\{\hat{Y}_t;\ t \in [0,\infty)\}$, denote optional projections of the processes $\{Z_t;\ t \in [0,\infty)\}$, $\{Y_t;\ t \in [0,\infty)\}$, onto the filtration $\{\mathcal{G}_t,\ t \in [0,\infty)\}$, then show that

$$\hat{Z}_t - \int_0^t \hat{Y}_s\, ds, \qquad \forall\, t \in [0,\infty),$$

is a $\{\mathcal{G}_t\}$-martingale.

Problem 4.8.7 Suppose that $\{(X(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ and $\{(Y(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ are L^2-martingales on $(\Omega, \mathcal{F}, P)$, and write

$$\mathcal{F}^X \triangleq \sigma\{X(t),\ t \in [0,\infty)\}, \quad \mathcal{F}^Y \triangleq \sigma\{Y(t),\ t \in [0,\infty)\},$$

and

$$\mathcal{F}_t^{X,Y} \triangleq \sigma\{X(s), Y(s),\ s \in [0,t]\}, \quad \forall\, t \in [0,\infty).$$

(a) If the σ-algebras $\mathcal{F}_X$ and $\mathcal{F}_Y$ are P-independent, show that $\{(X(t)Y(t),\ \mathcal{F}_t^{X,Y}),\ t \in [0,\infty)\}$ is a martingale on $(\Omega, \mathcal{F}, P)$.
Hint: use Theorem 1.4.15(f).
(b) If, in addition, the martingales $\{(X(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ and $\{(Y(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ are continuous, use the result from (a) to prove that

$$P\{[X,Y](t) = 0, \quad \forall\, t \in [0,\infty)\} = 1.$$

Problem 4.8.8 Suppose that $\{(X(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ is a right-continuous supermartingale on $(\Omega, \mathcal{F}, P)$.
(a) Show that $\{(X(t), \mathcal{G}_t);\ t \in [0,\infty)\}$ is a right-continuous supermartingale, for $\mathcal{G}_t \triangleq \mathcal{F}_{t+}$, $\forall\, t \in [0,\infty)$.
Hint: Use Theorem 2.7.9.

(b) Suppose, in addition, that $X(t,\omega) \in [0,\infty)$, $\forall\, t \in [0,\infty)$, $\forall\, \omega \in \Omega$, and put

$$S_1(\omega) \triangleq \inf\{t \in [0,\infty):\ X(t,\omega) = 0\}, \quad \forall\, \omega \in \Omega,$$

$$S_2(\omega) \triangleq \inf\{t \in [0,\infty):\ X(t-,\omega) = 0\}, \quad \forall\, \omega \in \Omega,$$

and $T(\omega) \triangleq S_1(\omega) \wedge S_2(\omega)$, $\forall\, \omega \in \Omega$. Show that

$$P[\omega:\ T(\omega) < \infty, \ \text{ and } \ X(t,\omega) = 0 \ \ \forall\, t \ge T(\omega)] = P[T < \infty].$$

Hint: Observe that the T_n defined by

$$T_n(\omega) \triangleq \inf\{t \in [0,\infty) : \ X(t,\omega) < n^{-1}\}, \quad \forall\, \omega \in \Omega, \ \forall\, n = 1, 2, \ldots$$

are $\{\mathcal{G}_t\}$-stopping times, and use the result established in (a).

Problem 4.8.9 (a) Suppose that $\{(W(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ is a standard scalar Wiener process on $(\Omega, \mathcal{F}, P)$. For each $\lambda \in \mathbb{R}$ define

$$X_t^\lambda \triangleq \exp\left\{\lambda W(t) - \frac{\lambda^2 t}{2}\right\}, \quad \forall\, t \in [0,\infty). \tag{4.136}$$

Show that $\{(X_t^\lambda, \mathcal{F}_t);\ t \in [0,\infty)\}$ is a martingale for each $\lambda \in \mathbb{R}$.
Hint : use the moment-generating function for a Gaussian-distributed random variable.

(b) Using (a) and Theorem 4.3.3 show that

$$P\left[\max_{0 \le s \le t} W(s) \ge \alpha t\right] \le \exp\left(\frac{-\alpha^2 t}{2}\right),$$

for all $\alpha, t \in [0,\infty)$.

Problem 4.8.10 Suppose that $\{(W(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ is a scalar standard Wiener process on probability space $(\Omega, \mathcal{F}, P)$. For each $x \in \mathbb{R}$ define the process $\{W^x(t),\ t \in [0,\infty)\}$ by

$$W^x(t,\omega) \triangleq x + W(t,\omega), \qquad \forall\, t \in [0,\infty), \ \forall\, \omega \in \Omega. \tag{4.137}$$

If $a, b \in \mathbb{R}$ are constants with $a < b$, then, for each $x \in \mathbb{R}$, define

$$T^x(\omega) \triangleq \inf\{t \in [0,\infty) \ : \ W^x(t,\omega) \notin (a,b)\}, \qquad \forall\, \omega \in \Omega.$$

(a) Show that $E[T^x] < \infty$ for each $x \in \mathbb{R}$.
(b) Using (a) show that

$$E[T^x] = (b-x)(x-a), \qquad \forall\, x \in (a,b).$$

Problem 4.8.11 Suppose that $\{(X(t), \mathcal{F}_t);\ t \in [0,\infty)\}$ is a non-negative continuous martingale such that

$$\lim_{t\to\infty} X(t) = 0, \qquad \text{a.s.}$$

Show that for each $x \in (0,\infty)$

$$P\left[\sup_{s\in[0,\infty)} X(s) \ge x \;\middle|\; X(0)\right] = \min\left[1, \frac{X(0)}{x}\right], \qquad \text{a.s.}$$

Hint: Stop $\{X(t)\}$ at the $\{\mathcal{F}_t\}$-stopping time given by

$$T(\omega) \triangleq \inf\{t \in [0,\infty) \ : \ X(t,\omega) \ge x\},$$

and use Corollary 4.5.8.

Problem 4.8.12 Suppose that $X \in \mathbf{M}_{loc}^{c,0}(\{\mathcal{F}_t\}, P)$, S is a $\{\mathcal{F}_t\}$-stopping time, and there is a constant $a \in [0, \infty)$ such that

$$E[X^2(S \wedge T)] \leq a,$$

for each $\{\mathcal{F}_t\}$-stopping time T for which $P[T < \infty] = 1$. Show that $\{(X(t \wedge S), \mathcal{F}_t);\ t \in [0, \infty)\}$ is an L^2-bounded martingale.

Problem 4.8.13 (a) Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathcal{G} \subset \mathcal{F}$ is a sub σ-algebra, and $A \in \mathcal{G}$. The *trace* of $\mathcal{G}$ on the event A is defined by

$$A \cap \mathcal{G} \triangleq \{A \cap G \mid G \in \mathcal{G}\}.$$

Now suppose that T and U are $\{\mathcal{F}_t\}$-stopping times. Show that

$$\{T \leq U\} \cap \mathcal{F}_T = \{T \leq U\} \cap \mathcal{F}_{T \wedge U},$$

that is, $\mathcal{F}_T$ and $\mathcal{F}_{T \wedge U}$ have identical trace on the event $\{T \leq U\}$.

(b) Suppose that $\{(X(t), \mathcal{F}_t);\ t \in [0, \infty)\}$ is a right-continuous martingale, and T and U are $\{\mathcal{F}_t\}$-stopping times, with U uniformly bounded i.e. $U(\omega) \leq B$, all $\omega \in \Omega$, for some constant $B \in [0, \infty)$. Show that

$$E\,[X(U) \mid \mathcal{F}_T] = X(T \wedge U), \qquad a.s.$$

Hint: Use the result in (a) together with the optional sampling theorem for ordered stopping times (see Corollary 4.5.6) and the localization property of conditional expectations established in Problem 1.6.11(a)(b).

(c) Again with T a $\{\mathcal{F}_t\}$-stopping time, put

$$Y(t) \triangleq \xi[X(t) - X(t \wedge T)], \qquad t \in [0, \infty),$$

where ξ is uniformly bounded and $\mathcal{F}_T$-measurable. Show that $\{(Y(t), \mathcal{F}_t);\ t \in [0, \infty)\}$ is a martingale. Hint: use Lemma 4.5.1 and the result established in (b).

Problem 4.8.14 Suppose that $X \in \mathbf{M}_{loc}^{c,0}(\{\mathcal{F}_t\}, P)$.

(a) Show that

$$\{\sup_{t \in [0,\infty)} X_t < \infty\} \subset \{\lim_{t \to \infty} X_t \text{ exists in } \mathbb{R}\} \qquad \text{a.s.}$$

Hint: define the $\{\mathcal{F}_t\}$-stopping times T_n by

$$T_n \triangleq \inf\{t \in [0, \infty) \;:\; X_t \geq n\}, \qquad \forall\, n = 1, 2, \ldots$$

and use the fact that $\{(n - X_{t \wedge T_n},\ \mathcal{F}_t);\ t \in [0, \infty)\}$ is a continuous non-negative local martingale.
(b) Next, show that

$$\{\lim_{t\to\infty} X_t \text{ exists in } I\!\!R\} = \{\sup_{t\in[0,\infty)} X_t < \infty\} \cup \{\inf_{t\in[0,\infty)} X_t > -\infty\} \qquad \text{a.s.}$$

(c) Finally, conclude that

$$P[\lim_{t\to\infty} X_t = +\infty] = P[\lim_{t\to\infty} X_t = -\infty] = 0,$$

and

$$\{\lim_{t\to\infty} X_t \text{ exists in } I\!\!R\} = \{\sup_{t\in[0,\infty)} |X_t| < \infty\} \qquad \text{a.s.}$$

Note: Recall the notation in Remark 1.4.17.

Problem 4.8.15 Suppose that $X \in \mathbf{M}^{c,0}_{loc}(\{\mathcal{F}_t\}, P)$.
(a) Define the $\{\mathcal{F}_t\}$-stopping time S by

$$S \triangleq \inf\{t \in [0, \infty)\ :\ [X](t) \geq a\},$$

for some constant $a \in (0, \infty)$, and show that $\{(X(t \wedge S), \mathcal{F}_t);\ t \in [0, \infty)\}$ is an L^2-bounded continuous martingale. Hint: use Problem 4.8.12.
(b) Next, show that

$$\{[X](\infty) < \infty\} \subset \{\lim_{t\to\infty} X_t \text{ exists in } I\!\!R\} \qquad \text{a.s.}$$

Hint: Use the result from (a) together with the $\{\mathcal{F}_t\}$-stopping times

$$S_n \triangleq \inf\{t \in [0, \infty)\ :\ [X](t) \geq n\}, \qquad \forall\, n = 1, 2, \ldots$$

(c) Next, conclude that

$$\{[X](\infty) < \infty\} = \{\lim_{t\to\infty} X_t \text{ exists in } I\!\!R\} \qquad \text{a.s.}$$

Hint: Use the $\{\mathcal{F}_t\}$-stopping times

$$T_n \triangleq \inf\{t \in [0, \infty)\ :\ |X_t| \geq n\}, \qquad \forall\, n = 1, 2, \ldots$$

to obtain the a.s. set-inclusion opposite to that established in (b).
Note: Recall the notation in Remark 1.4.17.

Problem 4.8.16 In this problem $\{\mathcal{F}_t,\ t \in [0, \infty)\}$ is a given *right-continuous* filtration on the probability space $(\Omega, \mathcal{F}, P)$, that is

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, \quad \text{for all } t \in [0, \infty).$$

(a) Suppose that T_n, $n = 1, 2, \ldots$, are $\{\mathcal{F}_t\}$-stopping times. Show that

$$T \triangleq \inf_n T_n \quad \text{is a } \{\mathcal{F}_t\}\text{-optional time,}$$

then conclude that it is a $\{\mathcal{F}_t\}$-stopping time.

Hint: Use Definition 3.3.1 and Proposition 3.3.3.

(b) Suppose that $\{(X(t), \mathcal{F}_t);\ t \in [0, \infty)\}$ is a right-continuous adapted process on $(\Omega, \mathcal{F}, P)$, and that T_n, $n = 1, 2, \ldots$, are $\{\mathcal{F}_t\}$-stopping times such that

(i) $0 \leq T_n(\omega) \leq T_{n+1}(\omega)$ for all $\omega \in \Omega$ and $n = 1, 2, \ldots$

(ii) $P\left[\lim_{n\to\infty} T_n = \infty\right] = 1$.

Establish the following: $\{(X_t, \mathcal{F}_t);\ t \in [0, \infty)\}$ is a local martingale if and only if the stopped processes $\{(X(t \wedge T_n), \mathcal{F}_t), t \in [0, \infty)\}$ are local martingales for each $n = 1, 2, \ldots$

Hint: If $\{(X(t \wedge T_n), \mathcal{F}_t), t \in [0, \infty)\}$ is a local martingale then (see Remark 4.6.2) it has a localizing sequence $\{S_{n,m}, m = 1, 2, \ldots\}$ of $\{\mathcal{F}_t\}$-stopping times. Show that, for each $n = 1, 2, \ldots$, there is is some positive integer $m(n)$ such that

$$P\left[S_{n,m(n)} < n \wedge T_n\right] < 2^{-n}.$$

Then put $\tilde{T}_n \triangleq T_n \wedge S_{n,m(n)}$, show that $\lim_{n\to\infty} \tilde{T}_n = \infty$ a.s., and use Corollary 4.5.8.

4.8.1 $\{(X_t, \mathcal{F}_t), t \in [0,\infty]\}$ is progress. measurable and $E[|X_T|] < \infty$, $E[X_T] = 0$ for every $\{\mathcal{F}_t\}$-stopping time T.

Fix $t \in [0,\infty)$ and fix $A \in \mathcal{F}_t$.

Define $T_1 : \Omega \longrightarrow [0,\infty]$ as

$$T_1(\omega) \triangleq t, \quad \forall \omega \in A$$
$$\triangleq \infty, \quad \forall \omega \notin A. \qquad (4.8.1)$$

Clearly T_1 is a $\{\mathcal{F}_t\}$-time, thus $E[|X_{T_1}|] < \infty$ and $E[X_{T_1}] = 0$. Thus

(1) $$0 = EX_T = \int_A X_t \, dP + \int_{A^c} X_\infty \, dP.$$

Moreover, since T_2 given by $T_2(\omega) \triangleq \infty$, $\forall \omega \in \Omega$, is trivially a $\{\mathcal{F}_t\}$-time, by hypothesis we have $E|X_\infty| < \infty$ and

$E X_\infty = 0$ thus

(2) $$0 = \int_A X_\infty \, dP + \int_{A^c} X_\infty \, dP$$

From (1), (2):

(3) $$E[X_t; A] = E[X_\infty; A], \quad \forall A \in \mathcal{F}_t$$

thus

(4) $$X_t = E[X_\infty \mid \mathcal{F}_t] \quad \text{a.s.}$$

It follows that $\{X_t, \ t \in [0,\infty]\}$ is u.i martingale.

$\{(X_t, \mathcal{F}_t),\ t \in [0,\infty)\}$ is a martingale on $(\Omega, \mathcal{F}, P)$. $t \to X_t(\omega)$ is right continuous on $[0,\infty)$ for each $\omega \in \Omega$.

Now let $s, t \in [0,\infty)$ with $s < t$, and let integer n_1 be such that

4.8.2

$$s + 1/n < t, \quad \forall\, n \geq n_1.$$

Then

$$E[X_t \mid \mathcal{F}_{s+1/n}] = X_{s+1/n} \quad \text{a.s.} \_\_\_ (1)$$

for each $n = 1+n_1, 2+n_1, 3+n_1, \ldots$

Now

$$\lim_{n\to\infty} X_{s+1/n} = X_s, \quad \forall\, \omega \in \Omega \_\_\_ (2)$$

From Theorem 2.7.8 one finds

$$\lim_{n\to\infty} E[X_t / \mathcal{F}_{s+1/n}]$$

$$= E\Big[X_t \,\Big|\, \bigcap_{n_1 \leq n < \infty} \mathcal{F}_{s+1/n}\Big] \text{ a.s.} \_\_\_ (3)$$

Now

$$\mathcal{F}_{s+} = \bigcap_{n_1 \le n < \infty} \mathcal{F}_{s+1/n} \qquad \text{———} \quad (4)$$

(see Proposition 3.1.28 (ii)).

Combining (1), (2), (3), (4):

4.8.2

$$E\left[X_t \mid \mathcal{F}_{s+} \right] = X_s \quad \text{a.s.}$$

showing that $\{(X_t, \mathcal{F}_{t+}), t \in [0,\infty)\}$ is a martingale on $(\Omega, \mathcal{F}, P)$.

(4.8.3)

Put $$Y_t \triangleq E[X_t | \mathcal{G}_t] \quad \text{---} \quad (1)$$

for each $t \in [0, \infty)$.

By Jensen :

$$E|Y_t| = E|E[X_t|\mathcal{G}_t]| \leq E[E\{|X_t| \,|\, \mathcal{G}_t\}]$$

thus

$$E|Y_t| \leq E|X_t| < \infty \quad \text{---} \quad (2)$$

Now fix $0 \leq s < t < \infty$. Then

$$E[Y_t | \mathcal{G}_s] = E[E[X_t|\mathcal{G}_t] | \mathcal{G}_s]$$

$$= E[X_t | \mathcal{G}_s] \qquad (\text{since } \mathcal{G}_s \subset \mathcal{G}_t)$$

$$= E[E[X_t | \mathcal{F}_s] | \mathcal{G}_s] \qquad (\mathcal{G}_s \subset \mathcal{F}_s)$$

$$= E[X_s | \mathcal{G}_s] \quad \text{a.s.} \quad \text{---} \quad (3)$$

(since $E[X_t | \mathcal{F}_s] = X_s$ a.s.)

From (3) :

$$E[Y_t | \mathcal{G}_s] = Y_s \quad \text{a.s.} \qquad \square$$

4.8.4

For each $n = 1, 2, \ldots$ put

$$T_n \triangleq \inf\{t \in [0,\infty) : |X_t| \geq n\} \quad \text{———} \quad (1)$$

i.e T_n is the début of the <u>closed</u> set $\{x \in \mathbb{R} : |x| \geq n\}$ by the <u>continuous</u> process $\{X_t\}$.

Since

$$\mathcal{F}_t^X \triangleq \sigma\{X_u,\ 0 \leq u \leq t\} \quad \text{———} \quad (2)$$

it follows that $\{(X_t, \mathcal{F}_t^X)\}$ is adapted, hence <u>Prop. 3.3.8 (b)</u> ensures that

$$T_n \text{ is a } \{\mathcal{F}_t^X\}\text{- stopping time} \quad \text{———} \quad (3)$$

Now $\{X_t\}$ is continuous and $\{(X_t, \mathcal{F}_t^X)\}$ is adapted,

i.e. Proposition 3.1.25 shows that $\{(X_t, \mathcal{F}_t^X)\}$ is actually progressively measurable, thus Proposition 3.3.14 shows that

$$X_{t\wedge T_n} \text{ is } \mathcal{F}^X_{t\wedge T_n}\text{-meas. for each } t\in[0,\infty). \quad \text{—— (4)}$$

Now fix $0\le s<t<\infty$:

By Proposition 4.6.9 and (1) we know that

$\{(X_{t\wedge T^n}, \mathcal{F}_t)\}$ is a martingale, thus

$$\mathbb{E}\left[X_{t\wedge T_n} \,\middle|\, \mathcal{F}_s\right] = X_{s\wedge T_n} \quad \text{a.s.} \quad \text{—— (5)}$$

Now $X_{s\wedge T_n}$ is $\mathcal{F}^X_{s\wedge T_n}$-meas. hence

is $\mathcal{F}^X_s$-meas (since $s\wedge T_n\le s$, thus $\mathcal{F}^X_{\cdot} \subset \mathcal{F}^X$)

and $\mathcal{F}_s^X \subset \mathcal{F}_s$ (by (2)).

Thus, conditioning each side of (5) w.r.t. $\mathcal{F}_s^X$, we get

$$E\left[X_{t \wedge T_n} \mid \mathcal{F}_s^X\right] = X_{s \wedge T_n} \quad \text{a.s.}$$

thus

$$\{(X_{t \wedge T_n}, \mathcal{F}_t^X)\} \text{ is a martingale}, \ \forall n = 1, 2, \ldots \qquad (6)$$

By (3), (6) and Remark 3.3.8 we get

$$X \in \underset{\sim}{M}_{loc}^{c}\left(\{\mathcal{F}_t^X\}, \underline{P}\right).$$

$X(t,\omega) \in \{a, b\}$.

Put

$$Y(t,\omega) \triangleq \frac{X(t,\omega) - a}{b - a}. \quad \text{—— ①}$$

Thus $Y(t,\omega) \in \{0, 1\}$ and $(Y(t), \mathcal{F}_t)$ is of course also a right-continuous martingale.

Note also that $Y^2(t,\omega) = Y(t,\omega)$, thus

$$E[Y^2(t)] = E[Y(t)] \quad \text{—— ②}$$

Now

$$E[(Y(t) - Y(0))^2]$$

$$= E[Y^2(t)] - 2E[Y(0)Y(t)] + E[Y^2(0)] \quad \text{—— ③}$$

and

$$E[Y(0) \cdot Y(t)] = E[E[Y(0) \cdot Y(t) \mid \mathcal{F}_0]]$$

$$= E[E[Y(t) \mid \mathcal{F}_0]\, Y(0)]$$

$$= E[Y^2(0)] \quad \text{—— ④}$$

By ②, ③ and ④ we have

$$E\left[(Y(t)-Y(0))^2\right] = E\left[Y^2(t)\right] - E\left[Y^2(0)\right]$$

4.8.5

$$= E[Y(t)] - E[Y(0)]$$

$$= 0 \quad \text{———} \quad (5)$$

(since $(Y(t), \mathcal{F}_t)$ is a martingale).

From (5)

$$P\left[Y(t) = Y(0)\right] = 1 \qquad \forall t \in [0, \infty) \quad \text{——} \quad (6)$$

Put

$$N_t \triangleq \{Y(t) \neq Y(0)\}$$

$$N \triangleq \bigcup_{t \in \mathbb{Q}_+} N_t .$$

In view of (6) and countability of $\mathbb{Q}_+$, we have

$$P(N) = 0.$$

Now, for $\omega \notin N$, we have

$$Y(t, \omega) = Y(0, \omega), \qquad \forall t \in \mathbb{Q}_+ \quad \text{——} \quad (7)$$

But $t \longrightarrow Y(t, \omega)$ is right-continuous, hence for each $\omega \notin N$, we find by 7

$$Y(t,\omega) = Y(0,\omega) \qquad \forall t \in [0,\infty) \quad \text{—— } (8)$$

In view of 1 and 8 we have

$$X(t,\omega) = X(0,\omega) \qquad \forall t \in [0,\infty)$$

for each $\omega \notin N$.

suppose first that $\{(X(t), \mathcal{F}_t)\}$ and $\{(Y(t), \mathcal{F}_t)\}$ are L_2-marts. with $\mathcal{F}^X$ and $\mathcal{F}^Y$ indep. for

4.8.7

$$\mathcal{F}^X \triangleq \sigma\{X(u),\ u \in [0,\infty)\}$$

$$\mathcal{F}^Y \triangleq \sigma\{Y(u),\ \text{"}\quad \text{"}\}$$

For each $t \in [0,\infty)$ put:

$$\mathcal{F}_t^X \triangleq \sigma\{X(u);\ u \in [0,t]\}$$

$$\mathcal{F}_t^Y \triangleq \sigma\{Y(u);\ \text{"}\quad \text{"}\}$$

$$\mathcal{F}_t^{XY} \triangleq \sigma\{X(u), Y(u);\ u \in [0,t]\}.$$

i) Fix $0 \le s < t < \infty$. We will show

① $E[X(t) \cdot Y(t) \mid \mathcal{F}_s^{XY}] = X(s)\, Y(s)$ a.s.

i.e. $\{(X(t)Y(t), \mathcal{F}_t^{XY})\}$ is a martingale.

In fact:

$$E\left[X(t).Y(t) \mid \mathcal{F}_s^{XY}\right] = E\left[X(t).Y(t) \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y\right]$$

$$= E\left[E\left[X(t).Y(t) \mid \mathcal{F}_s^X \vee \mathcal{F}_t^Y\right] \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y\right]$$

$$= E\left[Y(t).\, E\left[X(t) \mid \mathcal{F}_s^X \vee \mathcal{F}_t^Y\right] \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y\right]$$

$$= E\left[Y(t).\, E\left[X(t) \mid \mathcal{F}_s^X\right] \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y\right]$$

(by theorem 1.4.16 (f))

$$= E\left[X(t) \mid \mathcal{F}_s^X\right].\, E\left[Y(t) \mid \mathcal{F}_s^X \vee \mathcal{F}_s^Y\right]$$

$$= E\left[X(t) \mid \mathcal{F}_s^X\right] \cdot E\left[Y(t) \mid \mathcal{F}_s^Y\right] \quad a.s$$

(by theorem 1.4.16 (f))

$$= X(s)\, Y(s). \quad a.s.$$

This gives ①.

(b) Now put

$$\mathcal{N} \doteq \{ N \in \mathcal{F} : P(N) = 0 \}$$

and

$$\mathcal{G}_t \doteq \sigma \{ \mathcal{F}_t^{XY}, \mathcal{N} \}, \quad t \in [0, \infty) \quad \text{—} ②$$

We are given that

$$\left. \begin{aligned} X &\in \underset{\sim}{M}_2^c(\{\mathcal{F}_t\}, P) \\ Y &\in \underset{\sim}{M}_2^c(\{\mathcal{F}_t\}, P) \end{aligned} \right\} \quad \text{——} ③$$

Thus

$$\mathcal{F}_t^{XY} \subset \mathcal{F}_t, \quad \text{all } t \in [0, \infty)$$

and since $\mathcal{N} \subset \mathcal{F}_t$ " " ,

we must have

$$\mathcal{G}_t \subset \mathcal{F}_t, \quad t \in [0, +\infty) \quad \text{—} ④$$

Then, for $0 \leq s < t < \infty$, we have

$$E[X_t \mid \mathcal{G}_s] = E[\underbrace{E[X_t \mid \mathcal{F}_s]}_{X_s \text{ by } ③} \mid \mathcal{G}_s]$$

$$= E[X_s \mid \mathcal{G}_s]$$

$$= X_s \quad — (5)$$

(since X_s is $\mathcal{F}_s^X$ - meas, hence $\mathcal{G}_s$ - meas).

From (5)

$$X \in \underset{\sim}{M}_2^c(\{\mathcal{G}_t\}, \underline{P}) \quad — (6)$$

Similarly

$$Y \in \underset{\sim}{M}_2^c(\{\mathcal{G}_t\}, \underline{P}) \quad — (7)$$

also from (1) and (2)

$$E[X_t Y_t \mid \mathcal{G}_s] = E[X_t Y_t \mid \sigma\{\mathcal{F}_s^{XY}, \mathcal{N}\}]$$

$$= X_s Y_s \quad \text{a.s}$$

Thus

$$XY \in \underset{\sim}{M}^c(\{\mathcal{G}_t\}, \underline{P}) \quad — (8)$$

Now $\mathcal{G}_t$ includes all P- null events in $\mathcal{F}$.

Then, from (6), (7), (8) and uniqueness in Thrm 4.7.22 we get

$$\underline{P}\{[X,Y]_t = 0 \ , \ t \in [0,\infty)\} = 1.$$

4.8.8

(a) Fix $s, t \in [0, \infty)$ with $s < t$, and let integer n_1 be such that $s + 1/n < t$, $\forall n \geq n_1$. Since $(X(t), \mathcal{F}_t)$ is a supermartingale, we have

$$E[X_t \mid \mathcal{F}_{s+1/n}] \leq X(s+1/n) \quad \text{a.s.}$$
$$\forall n = 1+n_1, 2+n_1, \ldots \quad \text{—— (1)}$$

Now
$$\lim_{n \to \infty} X(s+1/n) = X(s), \quad \forall \omega \in \Omega \quad \text{—— (2)}$$

(since $\{X(t)\}$ is right-continuous process). also, from Theorem 2.7.9,

$$\lim_{n \to \infty} E[X(t) \mid \mathcal{F}_{s+1/n}]$$
$$= E[X(t) \mid \bigcap_{n_1 \leq n < \infty} \mathcal{F}_{s+1/n}] \quad \text{a.s.}$$
$$= E[X(t) \mid \mathcal{F}_{s+}] \quad \text{a.s.} \quad \text{—— (3)}$$

(last equality by Proposition 3.1.30 (ii)). Now combine (1), (2), (3) to get, upon taking $n \to \infty$ in (1):

$$E\left[X(t) \mid \mathcal{F}_{s+}\right] \leq X(s) \quad \text{a.s}$$

or

$$E\left[X(t) \mid \mathcal{G}_s\right] \leq X(s) \quad \text{a.s.}$$

This shows that $(X(t), \mathcal{G}_t)$ is a supermartingale.

<u>Note</u> here we did not need the fact that $X(t) \in [0, \infty)$. However this will be required in part (b) which follows.

(b) Write (4.8.8) 3

(4.1) $$T_n(\omega) \triangleq \inf\{t \in [0,\infty) : X(t,\omega) < \tfrac{1}{n}\}$$

$n = 1, 2, 3, \ldots$ note!

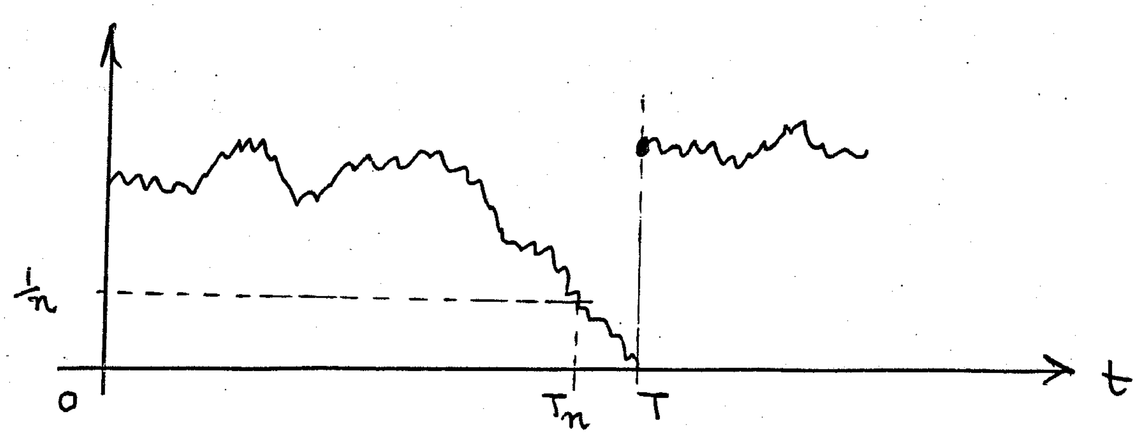
By Proposition 3.3.7 (a) one sees that T_n is an $\{\mathcal{F}_t\}$ - optional time (X is right - continuous and T_n is the début of the open set $(-\infty, 1/n)$). Thus T_n is a $\{\mathcal{G}_t\}$ - stopping time (by Proposition 3.3.3 (b)).

Moreover from (4.1) we have

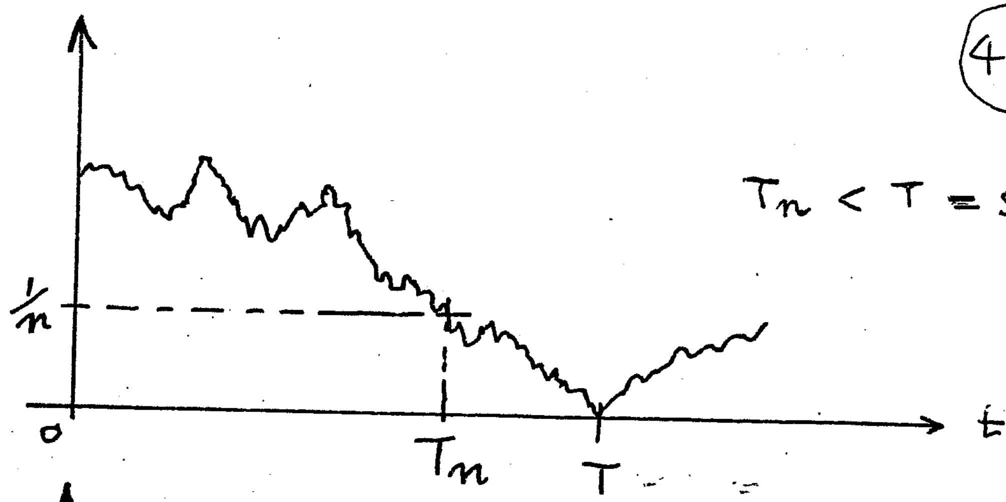
(4.2) $$T_n(\omega) \le T(\omega)$$

$n = 1, 2, \ldots, \quad \omega \in \Omega$

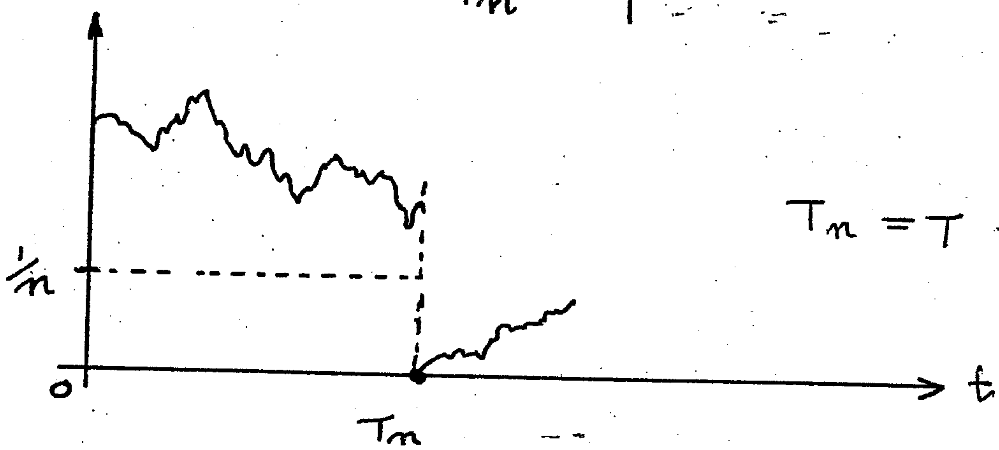
(see following diagrams which show the only possible cases)



$T_n < T = S_2 < S_1$ here



$T_n < T = S_1 = S_2$ here



$T_n = T = S_1 < S_2$ here

Now fix some $t \in [0, \infty)$. Since $X(t) \geq 0$ and (4.2) ensures $\{T \leq t\} \subset \{T_n \leq t\}$, we clearly have:

$$(4.3)\quad E[X(t)\,;\, T \leq t] \leq E[X(t)\,;\, T_n \leq t]$$

$$\leq E\big[\, E[X(t) \mid \mathcal{G}_{t \wedge T_n}]\,;\, T_n \leq t\big]$$

$$(\text{since } \{T_n \leq t\} \in \mathcal{G}_{t \wedge T_n})$$

$$\leq E[X(t \wedge T_n)\,;\, T_n \leq t]$$

The final inequality of (4.3) follows because $t \wedge T_n$ is a $\{\mathcal{G}_t\}$-stopping time (see Proposition 3.3.13(a)); since $(t \wedge T_n) \le t$, one sees from part (a) and Corollary 4.5.6 that

$$E[X(t) \mid \mathcal{G}_{t \wedge T_n}] \le X(t \wedge T_n) \quad \text{a.s.}$$

as required for the final inequality of (4.3).

From (4.3) we trivially get that

$$\text{(4.4)} \quad E[X(t); T \le t]$$

$$\le E[X(t \wedge T_n); T_n \le t]$$

$$\le E[X(T_n); T_n \le t] \quad (\text{since } t \wedge T_n \equiv T_n \text{ on } \{T_n \le t\})$$

$$\le 1/n.$$

where final inequality of (4.4) results from (4.1) which implies

$$0 \le X(T_n(\omega), \omega) \le \tfrac{1}{n}, \quad \forall \omega \in \{T_n \le t\}.$$

By (4.4) and $n \longrightarrow \infty$:

$$(4.5) \qquad E\left[X(t)\, I\{t \geq T\} \right] = 0, \quad \forall t \in [0,\infty).$$

Now let

$$(4.6) \qquad N_t \triangleq \left\{ \omega : X(t,\omega)\, I\{t \geq T(\omega)\} \neq 0 \right\}$$

$\forall t \in [0,\infty)$, and put

$$(4.7) \qquad N \triangleq \bigcup_{t \in Q_+} N_t .$$

Since $X(t) \geq 0$, one sees from (4.5) that

$\underline{P}(N_t) = 0$, hence

$$(4.8) \qquad \underline{P}(N) = 0.$$

Now clearly

$$\left[\{T < \infty\} - N \right] \subset \{ T < \infty \text{ and } X(t) = 0 \;\; \forall t \in [T,\infty) \cap Q_+ \} \subset \{T < \infty\}$$

so that (4.8) gives

$$\underline{P}\{ T < \infty : \; X(t) = 0, \; \forall t \in [T,\infty) \cap Q_+ \}$$

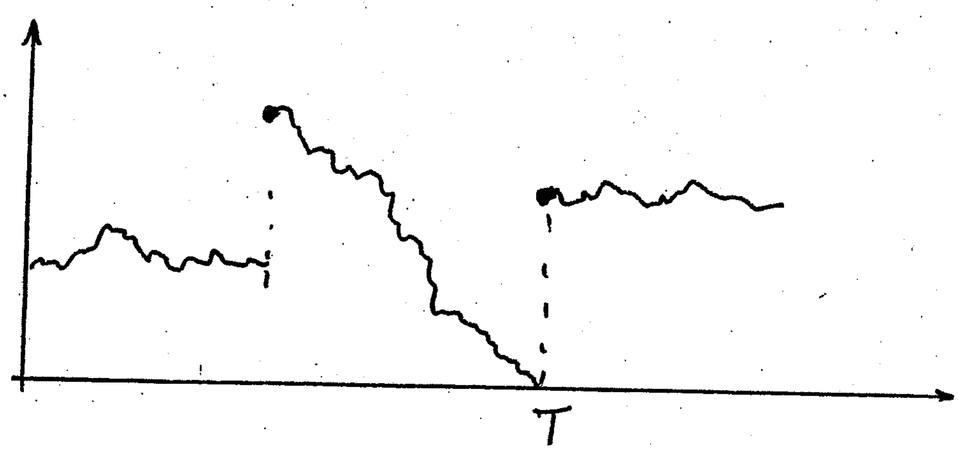
$$= \underline{P}[T < \infty].$$

By right-continuous sample-paths of X we then get

$$\mathbb{P}\{T < \infty : X(t) = 0, \ \forall t \in [T, \infty)\}$$

$$= \mathbb{P}\{T < \infty\}.$$

Observe that this result effectively eliminates sample-paths of the form:



(a) $\{(W_t, \mathcal{F}_t)\}$ is a scalar Wiener process on $(\Omega, \mathcal{F}, P)$. Fix $\lambda \in \mathbb{R}$ and put

(1) $$X_t^\lambda \triangleq \exp\{\lambda . W_t - \tfrac{1}{2}.\lambda^2 t\}, \quad \forall t \in (0,\infty)$$

Fix $0 \leq t < u < \infty$.

Then $(W_u - W_t) \sim N(0, u-t)$

thus

(2) $$E\left[\exp\{\lambda(W_u - W_t)\}\right] = \exp\left[\frac{\lambda^2(u-t)}{2}\right]$$

Now $(W_u - W_t) \perp\!\!\!\perp \mathcal{F}_t$, thus

(3) $$E\left[\exp\{\lambda(W_u - W_t)\} \,\middle|\, \mathcal{F}_t\right]$$

$$= E\left[\exp\{\lambda(W_u - W_t)\}\right] \quad \text{a.s.}$$

$$= \exp\left[\frac{\lambda^2(u-t)}{2}\right] \quad \text{a.s}$$

From (1) and (3)

4.8.9

$$E\left[X_u^\lambda \mid \mathcal{F}_t\right] = X_t^\lambda \qquad \text{a.s.}$$

(b) Fix $t \in [0,\infty)$ and $\alpha, \lambda \in (0,\infty)$

then clearly

(4) $\left\{\max_{s\le t} W_s \ge \alpha t\right\}$

$$= \left\{\max_{s\le t} \exp(\lambda W_s) \ge \exp(\lambda\alpha t)\right\}$$

and for each $s \in [0,t]$:

(5) $X_s^\lambda \ge \exp(\lambda W_s) \cdot \exp\left(-\frac{1}{2}\lambda^2 t\right)$.

(since $\lambda > 0$)

From (4) and (5):

(6) $\left\{\max_{s\le t} W_s \ge \alpha t\right\}$

$$\subset \left\{\max_{s\le t} X_s^\lambda \ge \exp\left(\lambda\alpha t - \frac{1}{2}\lambda^2 t\right)\right\}$$

thus

7) $P\left\{\max_{s\le t} W_s \ge \alpha t\right\}$

$$\leq P\left\{\max_{s\leq t} X_s^\lambda \geq \exp(\lambda\alpha t - \tfrac{1}{2}\lambda^2 t)\right\}$$

4.8.9

$$\leq E[X_t^\lambda] \cdot \exp(-\lambda\alpha t + \tfrac{1}{2}\lambda^2 t)$$

(using Theorem 4.3.2 with (a)).

Since $\{(X_t^\lambda, \mathcal{F}_t)\}$ is a martingale one has

(8) $$E[X_t^\lambda] = E[X_0^\lambda] = 1.$$

Combining (7) and (8):

(9) $$P\left\{\max_{s\leq t} W_s \geq \alpha t\right\}$$

$$\leq \exp(-\lambda\alpha t + \tfrac{1}{2}\lambda^2 t).$$

Now (9) holds for arbitrary $\lambda > 0$.

By simple calculus one finds

(10) $$\exp(-\lambda\alpha t + \tfrac{1}{2}\lambda^2 t) \geq \exp\left(-\frac{\alpha^2 t}{2}\right)$$

$\forall\, \lambda \in (0, \infty)$. From (9) and (10): (4.8.9)

(11) $$\mathbb{P}\left\{\max_{s \le t} W_s \ge \alpha t\right\} \le \exp\left(-\frac{\alpha^2 t}{2}\right).$$

Now fix $\beta \in (0, \infty)$.

We have shown that (11) holds for arbitrary $\alpha \in (0, \infty)$. Then let

$$\alpha \triangleq \frac{\beta}{t}$$

in (11). Then

(12) $$\mathbb{P}\left\{\max_{s \le t} W_s \ge \beta\right\} \le \exp\left(\frac{-\beta^2}{2t}\right).$$

(a) For each $x \in \mathbb{R}$, $t \in [0, \infty)$, $\omega \in \Omega$, put

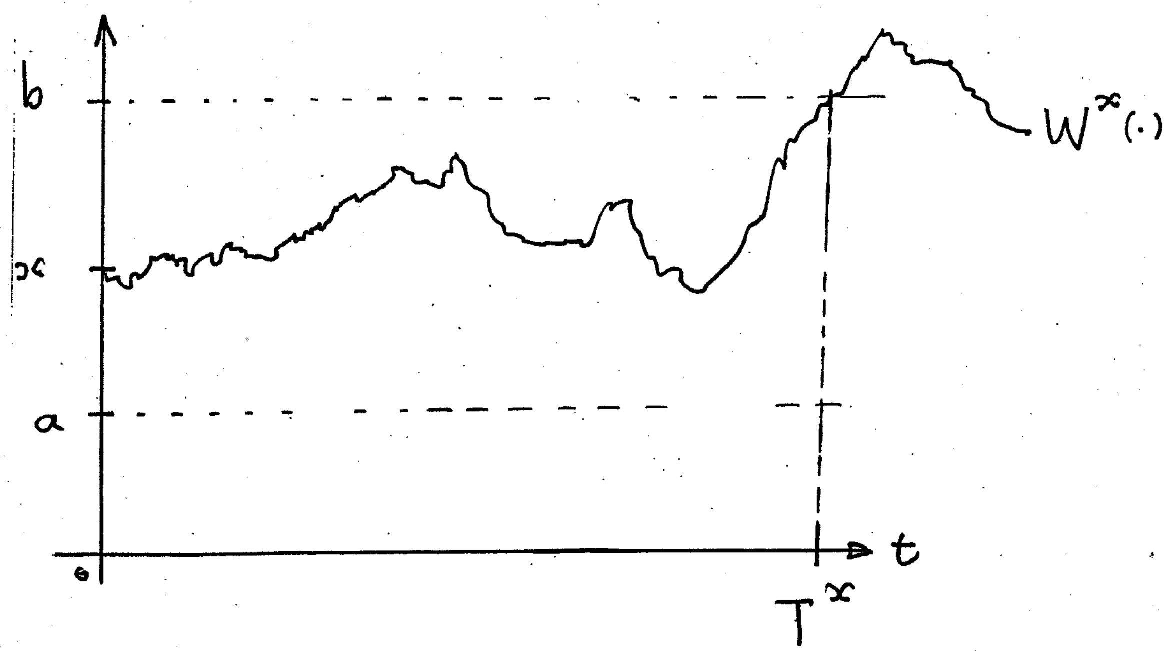
$$W^x(t) \triangleq x + W(t) \qquad (1)$$

where $\{(W(t), \mathcal{F}_t)\}$ is scalar standard Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$.

Put

$$T^x \triangleq \inf \{ t \in [0, \infty) : W^x(t) \notin (a, b) \} \qquad (2)$$

4.8.10



Since $W \in M_2^{c,0}(\{\mathcal{F}_t\}, \mathbb{P})$ (by Remark 4.1.9) we see from (1) that

$$W^x \in M_2^c(\{\mathcal{F}_t\}, \mathbb{P}) \qquad (3)$$

for each $x \in \mathbb{R}$. Moreover, from (1):

$$(W^x(t))^2 - t = [x + 2xW(t) + W^2(t)] - t$$

$$\equiv x + 2x W_t + (W_t^2 - t) \_\_\_\_ (4)$$

4.8.10

Clearly $xW \in M_{2,}^{c,0}(\{\mathcal{F}_t\}, P)$ and $\{(W_t^2 - t, \mathcal{F}_t)\}$ is a continuous martingale (by Proposition 4.1.8). Thus, if one defines

$$X_t^x \triangleq (W_t^x)^2 - t \_\_\_\_ (6)$$

then it follows from (4) that

$$X^x \in M^c(\{\mathcal{F}_t\}, P) \_\_\_\_ (7)$$

for each $x \in \mathbb{R}$. In view of (7) and Corollary 4.5.8 we see that

$\{(X^x(t \wedge T^x, \mathcal{F}_t), t \in [0, \infty)\}$ is a continuous martingale for each $x \in \mathbb{R}$.

Thus, for each $t \in [0, \infty)$ and $x \in \mathbb{R}$,

$$E[X^x(t \wedge T^x)] = E[X^x(0)] \equiv x^2 \_\_\_\_ (8)$$

for each $x \in \mathbb{R}$, or from (6): (see (6) + (1))

$$E\left[\left(W^{x}(t\wedge T^{x})\right)^2\right] = x^2 + E\left[t\wedge T^{x}\right] \quad (9)$$

for each $t \in [0,\infty)$, and $x \in \mathbb{R}$. 4.8.10

We must show $E[T^{x}] < \infty$, $\forall x \in \mathbb{R}$.

When $x \in [b,\infty)$ or $x \in (-\infty, a]$ then clearly $T^{x} \equiv 0$; thus suppose $x \in (a,b)$. Then of course

$$W^{x}(t\wedge T^{x}) \in [a,b], \quad \forall t \in [0,\infty)$$

thus

$$\left|W^{x}(t\wedge T^{x})\right| \leq |a| + |b|, \quad \forall t \in [0,\infty)$$

and hence

$$E\left[\left(W^{x}(t\wedge T^{x})\right)^2\right] \leq 2a^2 + 2b^2, \quad \forall t \in [0,\infty) \quad (10).$$

By (9) and (10):

$$E\left[t\wedge T^{x}\right] \leq 2a^2 + 2b^2 - x^2, \quad \forall t \in [0,\infty) \quad (11)$$

Now

4.8.10

$$E[T^x] = E[\lim_{n\to\infty} (n \wedge T^x)]$$

$$= \lim_{n\to\infty} E(n \wedge T^x) \quad \text{(by thm } \underline{1.2.11}\text{)}$$

hence, by (11):

$$E[T^x] \leq 2a^2 + 2b^2 - x^2, \quad \forall x \in (a,b).$$

Since $E[T^x] = 0 \quad \forall x \notin (a,b)$

we have $E[T^x] < \infty, \quad \forall x \in \mathbb{R}.$

b) Put

$$N \triangleq \{T^x = +\infty\} \tag{12}$$

In view of (a) we have

$$\underline{P}(N) = 0 \tag{13}$$

and, from (2) & (12):

$$\Omega = \{T^x < \infty,\ W^x(T^x) = a\}$$

$$\cup \{T^x < \infty,\ W^x(T^x) = b\} \cup N \tag{14}$$

where events on rhs of (14) are clearly

disjoint.

(4.8.10)

By Problem 6 and fact that $E\,T^x < \infty$, we see that

$$E[W(T^x)] = 0 \quad \text{\_\_\_\_} \quad (15)$$

$$E[T^x] = E[W^2(T^x)] \quad \text{\_\_\_\_} \quad (16).$$

Now evaluate RHS of (16):

$$E[W^2(T^x)] = E[(W^x(T^x) - x)^2]$$

$$= E[(W^x(T^x))^2] - 2x\,E[W^x(T^x)] + x^2$$

$$= E[(W^x(T^x))^2] - x^2 \quad \text{\_\_\_\_} \quad (17)$$

(since (1) and (15) give that

$$E[W^x(T^x)] = x \,) \quad \text{\_\_\_\_} \quad (18)$$

From (13) and (14):

$$E[(W^x(T^x))^2] = a^2\, p + b^2\, q \quad \text{\_\_\_\_} \quad (19)$$

where

$$(20) \text{\_\_\_\_} \left\{ \begin{array}{l} p \triangleq \underline{P}\{T^x < \infty,\ W^x(T^x) = a\} \\ q \triangleq \underline{P}\{T^x < \infty,\ W^x(T^x) = b\} \end{array} \right.$$

Moreover, from (13) and (14) and (20):

$$p + q = 1 \qquad \text{---} \qquad (21)$$

Now determine p: From (18), (13) and (14):

$$x = E[W^x(T^x)]$$

$$= a \cdot p + b q \qquad \text{---} \qquad (22)$$

From (21) and (22):

$$p = \frac{b - x}{b - a}, \qquad q = \frac{x - a}{b - a} \qquad \text{---} \qquad (23)$$

From (16), (17), (19), (23):

$$E[T^x] = a^2 \left(\frac{b - x}{b - a}\right) + b^2 \left(\frac{x - a}{b - a}\right) - x^2$$

$$= (b - x)(x - a), \quad \forall x \in (a, b).$$

Since $X \in M^{c,0}_{loc}(\{\mathcal{F}_t\}, \mathbb{P})$, and S is a $\{\mathcal{F}_t\}$-stopping time, we have

4.8.12

$$X^S \in M^{c,0}_{loc}(\{\mathcal{F}_t\}, \mathbb{P}) \quad \text{——} \quad (1)$$

(why ?). Let $\{U_n, \ n = 1, 2, \dots\}$ be a localizing sequence for the continuous local mart. in (1).

Fix some $t \in [0, \infty)$. Then $(t \wedge U_n)$ is a $\{\mathcal{F}_t\}$-stopping time (why ?) with $\mathbb{P}[(t \wedge U_n) < \infty] = 1$, thus

$$E[X^2(S \wedge t \wedge U_n)] \leq a$$
$$\forall n = 1, 2, \dots$$

or, equivalently,

$$E[(X^S)^2(t \wedge U_n)] \leq a, \quad \forall n = 1, 2, \dots \quad \text{——} \quad (2)$$

From (2) we have that

4.8.12

$\{ X^s(t \wedge U_n),\ n = 1,2,3,\dots \}$ is a u.i. sequence; since $\{ U_n,\ n = 1,2,\dots \}$ is a localizing sequence for X^s, it then follows that $\{ (X^s(t), \mathcal{F}_t) ;\ t \in [0,\infty) \}$ is a martingale (why?), that is

$$\{ (X(t \wedge s), \mathcal{F}_t) ;\ t \in [0,\infty) \}$$

is a martingale. Moreover

$$E[X^2(s \wedge t)] \leq a, \qquad \forall t \in [0,\infty),$$

hence $\{ (X(s \wedge t), \mathcal{F}_t) ; t \in [0,\infty) \}$ is L_2-bounded.

$\square$

4.8.13

Given $X \in \underset{\sim}{M}^c(\{\mathcal{F}_t\}, P)$

stopping times T and U.

(a) show

$$\{T \le U\} \cap \mathcal{F}_T = \{T \le U\} \cap \mathcal{F}_{T\wedge U} \quad \text{——} \quad (1)$$

We have

$$\{T \le U\} \cap \mathcal{F}_T \triangleq \{A \cap \{T \le U\} \mid A \in \mathcal{F}_T\} \quad \text{—} \quad (2)$$

$$\{T \le U\} \cap \mathcal{F}_{T\wedge U} \triangleq \{A \cap \{T \le U\} \mid A \in \mathcal{F}_{T\wedge U}\} \quad \text{—} \quad (3)$$

Since $\mathcal{F}_{T\wedge U} \subset \mathcal{F}_U$ (Proposition 3.3.13 (b)) — (4)

we have (trivially)

$$\{T \le U\} \cap \mathcal{F}_{T\wedge U} \subset \{T \le U\} \cap \mathcal{F}_T \quad \text{——} \quad (5)$$

For the opposite set-inclusion fix $B \in \{T \le U\} \cap \mathcal{F}_T$. From (2)

$$B = A \cap \{T \le U\} \quad \text{for some } A \in \mathcal{F}_T \quad \text{—} \quad (6)$$

4.8.13

Then of course
$$B = A \cap \{T \le U\} \cap \{T \le U\}$$
$$= B_1 \cap \{T \le U\} \quad \text{—— (7)}$$

for
$$B_1 \triangleq A \cap \{T \le U\} \in \mathcal{F}_{T \wedge U} \quad \text{—— (8)}$$

by Prop. 3.3.13(e) & $A \in \mathcal{F}_T$

From (7) (8)
$$B \in \{T \le U\} \cap \mathcal{F}_{T \wedge U} \quad \text{—— (9)}$$

ie
$$\{T \le U\} \cap \mathcal{F}_T = \{T \le U\} \cap \mathcal{F}_{T \wedge U} \quad \text{—— (10)}$$

Now (1) follows from (10) and (5).

(b) Now suppose U is bounded.

then of course
$$X_U = X_{T \wedge U} \quad \text{on } \{U < T\} \quad \text{—— (11)}$$

and
$$\{U < T\} \in \mathcal{F}_{T \wedge U} \subset \mathcal{F}_T \quad \text{(by Prop. 3.3.13 (b)(d))} \text{ —— (12)}$$

4.8.13

From (11) (12) & Problem 1.6.11 (a)

$$E[X_U \mid \mathcal{F}_T] = E[X_{T\wedge U} \mid \mathcal{F}_T] \quad \text{a.s. on } \{U<T\} \text{ ― } (13)$$

By Prop. 3.3.15 see that $X_{T\wedge U}$ is $\mathcal{F}_{T\wedge U}$-meas. hence $\mathcal{F}_T$-meas, hence

$$E[X_{T\wedge U} \mid \mathcal{F}_T] = X_{T\wedge U} \quad \text{a.s} \text{ ― } (14)$$

Combine (14) & (13)

$$E[X_U \mid \mathcal{F}_T] = X_{T\wedge U} \quad \text{a.s. on } \{U<T\} \text{ ― } (15)$$

From Problem 1.6.11 (b) and (1)

$$E[X_U \mid \mathcal{F}_T] = E[X_U \mid \mathcal{F}_{T\wedge U}] \quad \text{a.s. on } \{T\le U\} \text{ ― } (16)$$

also $T\wedge U \le U$ (bounded stopping time) hence from Corollary 4.5.6

$$E[X_U \mid \mathcal{F}_{T\wedge U}] = X_{T\wedge U} \quad \text{a.s.} \text{ ― } (17)$$

Combine (17) & (16)

$$E[X_U \mid \mathcal{F}_T] = X_{T\wedge U} \quad \text{a.s. on } \{T \leq U\} \quad \text{—— (18)}$$

Combine (18) and (15):

$$E[X_U \mid \mathcal{F}_T] = X_{T\wedge U} \quad \text{a.s.} \quad \text{—— (19)}$$

(c) Put

$$Y_t \triangleq \xi\,[X_t - X_{t\wedge T}] \quad t \in [0,\infty) \quad \text{—— (20)}$$

where

$$\xi \text{ is } \mathcal{F}_T\text{-meas \& } \underline{\text{uniformly bounded}}. \quad \text{—— (21)}$$

Fix a <u>bounded</u> stopping time U. Then

$$E|Y_U| < \infty \quad (\text{trivially} - \xi \text{ is bounded}) \quad \text{—— (22)}$$

and

$$E[Y_U] \overset{(20)}{=} E[\xi\,(X_U - X_{T\wedge U})]$$

$$= E\big[E[\xi(X_U - X_{T\wedge U}) \mid \mathcal{F}_T]\big]$$

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$$= E\left[E[X_U - X_{T \wedge U} \mid \mathcal{F}_T]\, \xi \right] \quad (\xi \in \mathcal{F}_T)$$

$$= E\left[\underbrace{\{ E[X_U \mid \mathcal{F}_T] - X_{T \wedge U} \}}_{0 \text{ (from (6))}} \xi \right] \quad (X_{T \wedge U} \text{ is } \mathcal{F}_T\text{-meas})$$

$$= 0 \quad \text{—— (23).}$$

Now $\{Y_t\}$ is continuous hence progr. meas. Then, from (23) and Lemma 4.5.1,

$$Y \in \underset{\sim}{M}^c(\{\mathcal{F}_t\}, P).$$