2.8 Problems

Problem 2.8.1 (a) Suppose that $\{(X_n, \mathcal{F}_n); n = 0, 1, 2, \ldots\}$ is an adapted sequence on a probability space $(\Omega, \mathcal{F}, P)$ with $E|X_n| < \infty$, $\forall\, n = 0, 1, 2, \ldots$ Show that $\{X_n;\ n = 0, 1, 2 \ldots\}$ can be written as

$$X_n = X_0 + A_n + M_n, \qquad \forall\, n = 0, 1, 2, \ldots \tag{2.134}$$

where $\{(M_n, \mathcal{F}_n); n = 0, 1, 2, \ldots\}$ is a martingale with $M_0 \equiv 0$, $A_0 \equiv 0$, and A_n is $\mathcal{F}_{n-1}$-measurable for all $n = 1, 2, \ldots$ Also, show that this decomposition is unique in the sense that, if $\{(\tilde{M}_n, \mathcal{F}_n); n = 0, 1, 2, \ldots\}$ is a martingale with $\tilde{M}_0 \equiv 0$, $\{\tilde{A}_n;\ n = 0, 1, 2 \ldots\}$ is a process such that $\tilde{A}_0 \equiv 0$ and $\tilde{A}_n$ is $\mathcal{F}_{n-1}$-measurable for each $n = 1, 2, \ldots$, and

$$X_n = X_0 + \tilde{A}_n + \tilde{M}_n, \qquad \forall\, n = 0, 1, 2, \ldots$$

then

$$P[M_n = \tilde{M}_n] = 1 \quad \text{and} \quad P[A_n = \tilde{A}_n] = 1, \qquad \forall\, n = 0, 1, 2, \ldots$$

Hint: Observe that

$$X_n - X_0 = \sum_{k=1}^{n} \{X_k - E[X_k \mid \mathcal{F}_{k-1}]\} + \sum_{k=1}^{n} \{E[X_k \mid \mathcal{F}_{k-1}] - X_{k-1}\},$$

for all $n = 1, 2, \ldots$

(b) In the decomposition given by (2.134) show that

$$0 \le A_n \le A_{n+1} \quad \text{a.s.} \quad \forall\, n = 0, 1, 2, \ldots$$

if and only if $\{(X_n, \mathcal{F}_n); n = 0, 1, 2, \ldots\}$ is a submartingale.
(c) Now suppose that $\{(X_n, \mathcal{F}_n); n = 0, 1, 2, \ldots\}$ is a uniformly integrable submartingale. Show that $\{M_n;\ n = 0, 1, 2 \ldots\}$ and $\{A_n;\ n = 0, 1, 2 \ldots\}$ in (2.134) are uniformly integrable and $E[A_\infty] < \infty$, where $A_\infty \triangleq \liminf_{n\to\infty} A_n$.

Problem 2.8.2 Suppose that $\{X_n;\ n = 0, 1, 2 \ldots\}$ is a uniformly integrable sequence of nonnegative random variables on $(\Omega, \mathcal{F}, P)$. Establish the following:

$$\limsup_{n\to\infty} E[X_n] \le E[\limsup_{n\to\infty} X_n].$$

Hint: Establish the result supposing that the X_n are uniformly bounded: $0 \le X_n(\omega) \le c < \infty$, $\forall\, n = 0, 1, 2, \ldots$, $\forall\, \omega \in \Omega$. Now generalize to the case where $\{X_n;\ n = 0, 1, 2 \ldots\}$ is u.i.

Problem 2.8.3 (a) Suppose that $\{X_n;\ n = 0, 1, 2 \ldots\}$ is a sequence of independent random variables on $(\Omega, \mathcal{F}, P)$, such that $P[X_n = c] < 1$, $\forall\, n = 0, 1, 2, \ldots$, $\forall\, c \in I\!R$ i.e. each X_n is *not* a.s. equal to a constant value. Put

$$\mathcal{F}_n \triangleq \sigma\{X_0, X_1, \ldots X_n\},; \qquad \forall\, n = 0, 1, 2, \ldots$$

Show that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ with $\mathcal{F}_n \neq \mathcal{F}_{n+1}$ i.e. $\mathcal{F}_{n+1}$ is strictly larger than $\mathcal{F}_n$.

(b) In Proposition 2.1.12 we saw the following: suppose $\{\mathcal{F}_n,\ n = 0, 1, 2, \ldots\}$ is a given filtration in a probability space $(\Omega, \mathcal{F}, P)$, and S, T are stopping times with respect to $\{\mathcal{F}_n,\ n = 0, 1, 2, \ldots\}$. If $A \in \mathcal{F}_S$ then

$$A \cap \{S \leq T\} \in \mathcal{F}_{S \wedge T}. \tag{2.135}$$

Show by a counterexample that (2.135) is generally false when $A \in \mathcal{F}_T$.

✓ **Problem 2.8.4** Suppose that $\{\mathcal{F}_n,\ n = 0, 1, 2, \ldots\}$ is a filtration in the probability space $(\Omega, \mathcal{F}, P)$, and T is a $\{\mathcal{F}_n\}$-stopping time such that

$$P\,[T \leq n + 1 \,|\, \mathcal{F}_n] > a, \qquad \forall\, n = 0, 1, 2, \ldots$$

for some constant $a \in (0, 1)$.

(a) Show that

$$P[T > k] \leq (1 - a)^k, \qquad \forall\, k = 1, 2, \ldots$$

Hint: Observe that $P[T > k] = P[T > k,\ T > (k - 1)]$, $\forall\, k = 1, 2, \ldots$, and use induction.

(b) Use the result from (a) to conclude that $E[T] < \infty$.

✓ **Problem 2.8.5** (a) Suppose that Y is a *zero-mean* random variable on $(\Omega, \mathcal{F}, P)$, with $|Y(\omega)| \leq a$ for all $\omega \in \Omega$ and some constant $a \in [0, \infty)$. Show that

$$E \exp\{\lambda Y\} \leq \frac{e^{a\lambda} + e^{-a\lambda}}{2}, \qquad \forall\, \lambda \in I\!R.$$

Hint: Put $g(x) \triangleq \exp\{\lambda x\}$, $x \in [-a, a]$, and observe from convexity of $g(\cdot)$ that

$$g(x) \leq \frac{a - x}{2a} g(-a) + \frac{a + x}{2a} g(a), \qquad \forall\, x \in [-a, a].$$

(b) Suppose that $\{Y_k,\ k = 1, 2, \ldots\}$ are independent integrable random variables on $(\Omega, \mathcal{F}, P)$ such that $EY_k = 0$ and $|Y_k(\omega)| \leq a$ for all $\omega \in \Omega$ and $k = 1, 2, \ldots$ Put

$$X_n \triangleq \sum_{k=1}^{n} Y_k, \qquad \mathcal{F}_n \triangleq \sigma\{Y_1, Y_2, \ldots, Y_n\}, \qquad \forall n = 1, 2, \ldots$$

Use the result in (a) to establish that

$$P\left[\max_{1 \leq k \leq n} X_k \geq x\right] \leq \exp\left\{\frac{-x^2}{2a^2 n}\right\}, \qquad \forall\, x \in [0, \infty), \quad \forall n = 1, 2, \ldots.$$

Hint: Put $Z_n \triangleq e^{\lambda X_n}$ for arbitrary $\lambda \in (0, \infty)$, show that $\{(Z_n, \mathcal{F}_n); n = 0, 1, 2, \ldots\}$ is a non-negative submartingale, and apply Theorem 2.4.1. Then find the value of parameter λ which minimizes the resulting upper-bound. Note also the useful inequality $(e^x + e^{-x})/2 \leq \exp[x^2/2]$, $x \in I\!R$.

Problem 2.8.6 Let $\{X_n, n = 1, 2, \ldots\}$ be an i.i.d. sequence on $(\Omega, \mathcal{F}, P)$ with $E|X_n| < \infty$ and $\mu \triangleq E[X_n]$. Put

$$S_n \triangleq \sum_{i=1}^{n} X_i, \qquad \mathcal{F}_n \triangleq \sigma\{X_1, \ldots, X_n\}, \quad \forall\, n = 1, 2, \ldots$$

and let $T : \Omega \to \{1, 2, \ldots, \infty\}$ be an $\mathcal{F}_n$-stopping time (i.e. $\{T \leq n\} \in \mathcal{F}_n$, for all $n = 1, 2, \ldots$) such that $E[T] < \infty$. Establish the following:

(a) $E[|X_n|\, I\{n \leq T\}] = E[|X_1|]\, P\{T \geq n\}$ and $E[X_n I\{n \leq T\}] = \mu P\{T \geq n\}$, for all $n = 1, 2, \ldots$

(b) $E[S_T] = \mu E[T]$ (hint: use the result from (a) together with the monotone and dominated convergence theorems).

Problem 2.8.7 Suppose that $\{X_n;\, n = 0, 1, 2 \ldots\}$ is a sequence of independent and identically distributed r.v.'s on $(\Omega, \mathcal{F}, P)$. Put

$$\mu(\Gamma) \triangleq P(X_n \in \Gamma), \quad \forall\, \Gamma \in \mathcal{B}(I\!\!R),$$

and

$$\mathcal{F}_n \triangleq \sigma\{X_0, X_1, X_2, \ldots X_n\}$$

for all $n = 0, 1, 2, \ldots$ If $T : \Omega \to \{0, 1, 2, \ldots, +\infty\}$ is a stopping time with respect to $\{\mathcal{F}_n,\, n = 0, 1, 2, \ldots\}$ such that $\{T < \infty\} = \Omega$, establish the following: For any $A \in \mathcal{F}_T$, integer $m \geq 1$, and sets $\Gamma_n \in \mathcal{B}(I\!\!R)$, $\forall\, n = 1, 2, \ldots m$, we have

$$P(\omega \in A :\ X_{T(\omega)+n}(\omega) \in \Gamma_n,\ \forall\, n = 1, 2, \ldots m) = P(A) \prod_{n=1}^{m} \mu(\Gamma_n).$$

Problem 2.8.8 Suppose that $\{(X_n, \mathcal{F}_n); n = 0, 1, 2, \ldots\}$ is a supermartingale on $(\Omega, \mathcal{F}, P)$ with $X_n(\omega) \in [0, \infty)$, $\forall\, n = 0, 1, 2, \ldots$, $\forall \omega \in \Omega$, and put

$$T(\omega) \triangleq \min\{n : X_n(\omega) = 0\}, \quad \forall\, \omega \in \Omega.$$

Prove that

$$P[\omega : T(\omega) < \infty \quad \text{and} \quad X_n(\omega) = 0, \ \ n \geq T(\omega)] = P[T < \infty].$$

Problem 2.8.9 Suppose that $\{\mathcal{F}_n,\, n = 0, 1, 2, \ldots\}$ is a filtration in $(\Omega, \mathcal{F}, P)$, and $\{X_n,\ n = 0, 1, 2, \ldots\}$ is a sequence of random variables on $(\Omega, \mathcal{F}, P)$ converging a.s. to a random variable X. If $|X_n(\omega)| \leq Z(\omega)$, for all $n = 0, 1, 2 \ldots$ and $\omega \in \Omega$, for some random variable Z such that $E|Z| < \infty$, establish that

$$\lim_{n \to \infty} E\left[X_n \mid \mathcal{F}_n\right] = E\left[X \mid \mathcal{F}_\infty\right] \quad \text{a.s.}$$

Hint: Put $Y_m \triangleq \sup_{n \geq m} |X_n - X|$, for all $m = 0, 1, 2, \ldots$ and

$$\Delta_n \triangleq \left| E\left[X_n \mid \mathcal{F}_n\right] - E\left[X \mid \mathcal{F}_\infty\right] \right|, \quad n = 0, 1, 2, \ldots$$

and show that

$$\limsup_{n \to \infty} \Delta_n \leq E\left[Y_m \mid \mathcal{F}_\infty\right], \quad \text{for each } m.$$

Problem 2.8.10 Suppose that $\{X_n;\ n = 0, 1, 2 \ldots\}$ is a discrete-parameter process on $(\Omega, \mathcal{F}, P)$, let $\{\mathcal{F}_n,\ n = 0, 1, 2, \ldots\}$ be the raw filtration generated by X_n, namely

$$\mathcal{F}_n \triangleq \sigma\{X_0, X_1, X_2, \ldots, X_n\}, \qquad \forall\, n = 0, 1, 2, \ldots$$

and let $T : \Omega \to \{0, 1, 2, \ldots, +\infty\}$ be a $\{\mathcal{F}_n\}$-stopping time. Establish the following:
(a) If $\omega_1, \omega_2 \in \Omega$ are such that $T(\omega_1) < +\infty$, and $X_k(\omega_1) = X_k(\omega_2)$, $\forall\ k = 0, 1, 2, \ldots, T(\omega_1)$, then $T(\omega_1) = T(\omega_2)$.
(b) If $\omega_1, \omega_2 \in \Omega$ are such that $T(\omega_1) = +\infty$, and $X_k(\omega_1) = X_k(\omega_2)$, $\forall\ k = 0, 1, 2, \ldots$, then $T(\omega_2) = +\infty$.
(c) For each $n = 0, 1, 2, \ldots$, there exists a set $\Gamma_n \in \mathcal{B}(I\!R^{n+1})$ such that

$$\{\omega \in \Omega\ : T(\omega) = n\} = \{\omega \in \Omega\ : (X_0(\omega), X_1(\omega), X_2(\omega), \ldots, X_n(\omega)) \in \Gamma_n\}, \qquad \forall\, n = 0, 1, 2, \ldots,$$

and, if C_n is defined by

$$C_n \triangleq \{(x_0, x_1, x_2, \ldots) \in I\!R^\infty\ :\ (x_0, x_1, x_2, \ldots, x_n) \in \Gamma_n\}, \qquad \forall\, n = 0, 1, 2, \ldots,$$

then $C_m \cap C_n = \emptyset$ when $m \neq n$.
(d) For each $n = 0, 1, 2, \ldots$ one has

$$\{T = n\} = \{(X_{0\wedge T}, X_{1\wedge T}, \ldots, X_{n\wedge T}) \in \Gamma_n\},$$

where Γ_n is the set constructed in (c).

Now use (d) to establish the following characterization of the pre σ-algebra generated by the stopping time T:

(2.136) $$\mathcal{F}_T = \sigma\{X_{k\wedge T},\ k = 0, 1, 2, \ldots\}.$$

Briefly explain the intuitive significance of the results in 7(a), 7(b) and (2.136).
Hint : use Proposition 1.3.19 for parts (a), (b), (c), and Theorem 1.3.22 for part (d).

(a) $\{(X_n, \mathcal{F}_n)\}$ adapted, $E|X_n| < \infty$, $\forall n$.

Now define

2.8.1

$$(1.1) \quad \begin{cases} A_0 \triangleq 0 \\ A_n \triangleq \sum_{k=1}^{n} E[X_k - X_{k-1} \mid \mathcal{F}_{k-1}] \quad \forall n = 1, 2 \dots \end{cases}$$

and

$$(1.2) \quad M_n \triangleq (X_n - X_0) - A_n, \quad \forall n = 0, 1 \dots$$

Clearly

$$(1.3) \quad M_0 \equiv 0$$

and

$$(1.4) \quad E[M_1 \mid \mathcal{F}_0] = E[X_1 - X_0 \mid \mathcal{F}_0] - E[A_1 \mid \mathcal{F}_0]$$

$$= E[X_1 - X_0 \mid \mathcal{F}_0] - E[E[X_1 - X_0 \mid \mathcal{F}_0] \mid \mathcal{F}_0]$$

$$= 0 \equiv M_0.$$

When $n = 1, 2 \dots$ then

$$(1.5) \quad E[M_{n+1} \mid \mathcal{F}_n] = E[X_{n+1} - X_0 \mid \mathcal{F}_n]$$

$$- E\left[\sum_{k=1}^{n} E[X_k - X_{k-1} \mid \mathcal{F}_{k-1}] \mid \mathcal{F}_n\right]$$

$$- E\left[X_{n+1} - X_n \,\middle|\, \mathcal{F}_n\right] \qquad 2.8.1$$

$$= [X_n - X_0] - \underbrace{\sum_{k=1}^{n} E\left[X_k - X_{k-1} \,\middle|\, \mathcal{F}_{k-1}\right]}_{A_n}$$

$$= M_n \qquad \text{(see (1.1) \& (1.2))}$$

thus $\{(M_n, \mathcal{F}_n),\ n = 0, 1, 2 \dots\}$ is a martingale. also observe from (1.1) that A_n is $\mathcal{F}_{n-1}$ - meas. for all $n = 1, 2 \dots$

By (1.2) we get

(1.6) $$X_n = X_0 + M_n + A_n, \qquad n = 0, 1, 2 \dots$$

as required.

For uniqueness, suppose

(1.7) $$X_n = X_0 + \overline{M}_n + \overline{A}_n, \qquad n = 0, 1, 2 \dots$$

By (1.7) & (1.6):

2.8.1

$$M_n - \overline{M}_n = \overline{A}_n - A_n \qquad n = 0,1,2\ldots$$

thus $\{(\overline{A}_n - A_n, \mathcal{F}_n),\ n = 0,1,2\ldots\}$ is a martingale, thus

$$\underbrace{E[\overline{A}_n - A_n \mid \mathcal{F}_{n-1}]}_{\overline{A}_n - A_n \ \text{a.s.}} = \overline{A}_{n-1} - A_{n-1} \quad \text{a.s.}$$

(since $\overline{A}_n - A_n$ is $\mathcal{F}_{n-1}$-meas.)

Thus $\overline{A}_n = A_n$ a.s., $\forall n = 1,2\ldots$

and so $\overline{M}_n = M_n$ a.s. $\forall n = 1,2\ldots$

(b) suppose $\{(X_n, \mathcal{F}_n)\}$ is a submartingale.

then $E[X_k - X_{k-1} \mid \mathcal{F}_{k-1}] \geq 0$ a.s.

thus $0 \leq A_n \leq A_{n+1}$ a.s.

(see (1.1)).

Conversely, when $0 \leq A_n \leq A_{n+1}$ a.s. then, from (1.1), 2.8.1

$$E[X_{n+1} - X_n \mid \mathcal{F}_n] \geq 0 \quad \text{a.s.}$$

thus $\{(X_n, \mathcal{F}_n)\}$ is a submartingale.

(c) $\{(X_n, \mathcal{F}_n)\}$ is a u.i submart, it is L_1-bounded

Now by (1.2)

$$A_n = (X_n - X_0) - M_n$$

thus

$$E A_n = E X_n - E X_0 - \underbrace{E M_n}_{\substack{\| \\ E M_0 = 0}}$$

thus

$$E A_n \leq E|X_n| + E|X_0|.$$

Since $A_n \leq A_{n+1}$ a.s. $\forall n$, Fatou thrm gives (with $A_\infty \triangleq \liminf\limits_{n\to\infty} A_n$)

$$E A_\infty \leq \sup_n E|X_n| + E|X_0| < \infty$$

Since $0 \leq A_n \leq A_\infty$ a.s. clearly $\{A_n, n = 0, 1, 2 \ldots\}$ is u.i.

2.8.2

a) we have

(2.1) $$0 \leq X_n(\omega) \leq c < \infty$$

$\forall \omega \in \Omega$, $\forall n = 1, 2, 3 \ldots$ thus

(2.2) $$0 \leq c - X_n(\omega)$$

$\forall \omega \in \Omega$, $\forall n = 1, 2, 3 \ldots$

By (2.2) and Fatou lemma:

(2.3) $$\liminf_{n \to \infty} E[c - X_n]$$

$$\geq E[\liminf_{n \to \infty} (c - X_n)]$$

Now, for any sequence of numbers $\{x_n\}$ one has

(2.4) $$\liminf_{n \to \infty} (c - x_n)$$

$$= c - \limsup_{n \to \infty} x_n$$

By (2.3) and (2.4): 2.8.2

$$c - \limsup_{n\to\infty} E[X_n]$$

$$\geq E[c - \limsup_{n\to\infty} X_n]$$

$$\geq c - E[\limsup_{n\to\infty} X_n]$$

so that

$$\limsup_{n\to\infty} E[X_n] \leq E[\limsup_{n\to\infty} X_n]$$

as required.

b) Now suppose $\{X_n\}$ is u.i and non-negative. Fix some arbitrary $\varepsilon > 0$. Then $\exists$ some $c_\varepsilon \in (0, \infty)$ s.t.

$$E[X_n I\{X_n > c_\varepsilon\}] < \varepsilon \qquad (2.5)$$

$\forall\, n = 0, 1, 2 \ldots$

Now clearly 2.8.2

(2.6) $0 \leq X_n I\{X_n \leq c_\varepsilon\} \leq c_\varepsilon < \infty$

$\forall n = 0, 1, 2 \ldots \quad \forall \omega \in \Omega.$

By (2.6) and result in (a) we get

(2.7) $$\limsup_{n \to \infty} E[X_n I\{X_n \leq c_\varepsilon\}]$$

$$\leq E[\limsup_{n \to \infty} X_n I\{X_n \leq c_\varepsilon\}]$$

from part (a)

$$\leq E[\limsup_{n \to \infty} X_n].$$

obvious, since $X_n I\{X_n \leq c_\varepsilon\} \leq X_n$.

Now of course

$$E[X_n] = E[X_n I\{X_n \le c_\varepsilon\}] + E[X_n I\{X_n > c_\varepsilon\}] \tag{2.8}$$

For sequences of real numbers $\{x_n\}$ and $\{y_n\}$:

$$\limsup_{n\to\infty} (x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \tag{2.9}$$

Then from (2.8) and (2.9)

$$\limsup_{n\to\infty} E[X_n] \le \limsup_{n\to\infty} E[X_n I\{X_n \le c_\varepsilon\}] + \limsup_{n\to\infty} E[X_n I\{X_n > c_\varepsilon\}] \tag{2.10}$$

2.8.2

$$\leq E[\limsup_{n\to\infty} X_n] + \varepsilon$$

by (2.7) and (2.5)

Since $\varepsilon > 0$ in (2.10) is arbitrary we get

$$\limsup_{n\to\infty} E[X_n]$$

$$\leq E[\limsup X_n]$$

as required.

(a) First construct a filtration $\{\mathcal{F}_n, n = 0, 1, 2 \ldots\}$ in $(\Omega, \mathcal{F}, P)$ such that

2.8.3

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \quad \text{and} \quad \mathcal{F}_n \neq \mathcal{F}_{n+1} \tag{3.1}$$

for each $n = 0, 1, 2 \ldots$

This can be done e.g. by letting $\{X_n, n = 0, 1, 2 \ldots\}$ be some independent sequence on $(\Omega, \mathcal{F}, P)$ and writing

$$\mathcal{F}_n \triangleq \sigma\{X_0, X_1 \ldots X_n\} \tag{3.2}$$

for each $n = 0, 1, 2 \ldots$

Clearly $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $\forall n = 0, 1, 2 \ldots$

To see the second part of (3.1) suppose the contrary, namely

2.8.3

$$\mathcal{F}_{n+1} = \mathcal{F}_n \tag{3.3}$$

for some non-negative integer n. Then X_{n+1} is meas. w.r.t. $\sigma\{X_0, X_1 \ldots X_n\}$. Thus, by Doob's theorem <u>1.3.21</u>, $\exists$ some $\mathcal{B}(\mathbb{R}^{n+1})$-meas. mapping $g : \mathbb{R}^{n+1} \to \bar{\mathbb{R}}$ such that

$$X_{n+1}(\omega) = g(X_0(\omega) \ldots X_n(\omega))$$

for all $\omega \in \Omega$, in contradiction to the postulated independence of $\{X_0, X_1 \ldots X_{n+1}\}$. Thus (3.1) follows.

(b) Fix integers $m > n \geq 0$ and choose

event $A \in \mathcal{F}_m$, $A \notin \mathcal{F}_n$. 2.8.3

(since $\mathcal{F}_n \subsetneqq \mathcal{F}_m$ we can make this choice, taking filtration $\{\mathcal{F}_n\}$ from 3 (a))

Put

$$\left.\begin{aligned} S(\omega) &\triangleq n \\ T(\omega) &\triangleq m \end{aligned}\right\} \quad \forall\, \omega \in \Omega$$

Then $A \in \mathcal{F}_T$.

If (0.1) held true for this event A then we have

$$A = A \cap \{S \leq T\} \quad (\text{since } \{S \leq T\} = \Omega)$$

$$\in \mathcal{F}_{S \wedge T} = \mathcal{F}_n$$

which contradicts $A \notin \mathcal{F}_n$.

(a) fix some $k = 1, 2, \ldots$ 2.8.4

Since $[T > k] \subset [T > k-1]$, clearly

$$P[T > k] = P[T > k, T > k-1]$$

$$= E\left[I\{T > k\} . I\{T > k-1\} \right]$$

$$= E\left[E\left[I\{T > k\} \middle| \mathcal{F}_{k-1} \right] . I\{T > k-1\} \right] \quad (1)$$

at the last equality in (1) we used $\{T > k-1\} \in \mathcal{F}_{k-1}$, which follows since T is a stopping time. Moreover,

$$E\left[I\{T > k\} \middle| \mathcal{F}_{k-1} \right] = P\left[T > k \middle| \mathcal{F}_{k-1} \right]$$

$$= 1 - P\left[T \leq k \middle| \mathcal{F}_{k-1} \right]$$

$$< (1 - a) \quad (2)$$

From (1) and (2):

$$P[T > k] < E\left[(1-a) I\{T > k-1\} \right]$$

2.8.4

hence

$$P[T > k] < (1-a)\,P[T > k-1] \quad \text{—— (3)}$$

Now $P[T > 0] \leq 1$ hence by induction & (3)

we get

$$P[T > k] < (1-a)^k, \quad k = 1, 2, \ldots \quad \text{—— (4)}$$

(b) observe that

$$\sum_{k=0}^{\infty} P[T > k] = \sum_{k=0}^{\infty} E[I\{T > k\}]$$

$$= E\left\{\sum_{k=0}^{\infty} I[T > k]\right\} \quad \text{—— (5)}$$

(by Monotone Convergence Theorem)

But clearly

$$T = \sum_{k=0}^{\infty} I[T > k] \quad \text{—— (6)}$$

By (5), (6)

2.8.4

$$E[T] = E\left[\sum_{k=0}^{\infty} I[T > k]\right]$$

$$= \sum_{k=0}^{\infty} P[T > k]$$

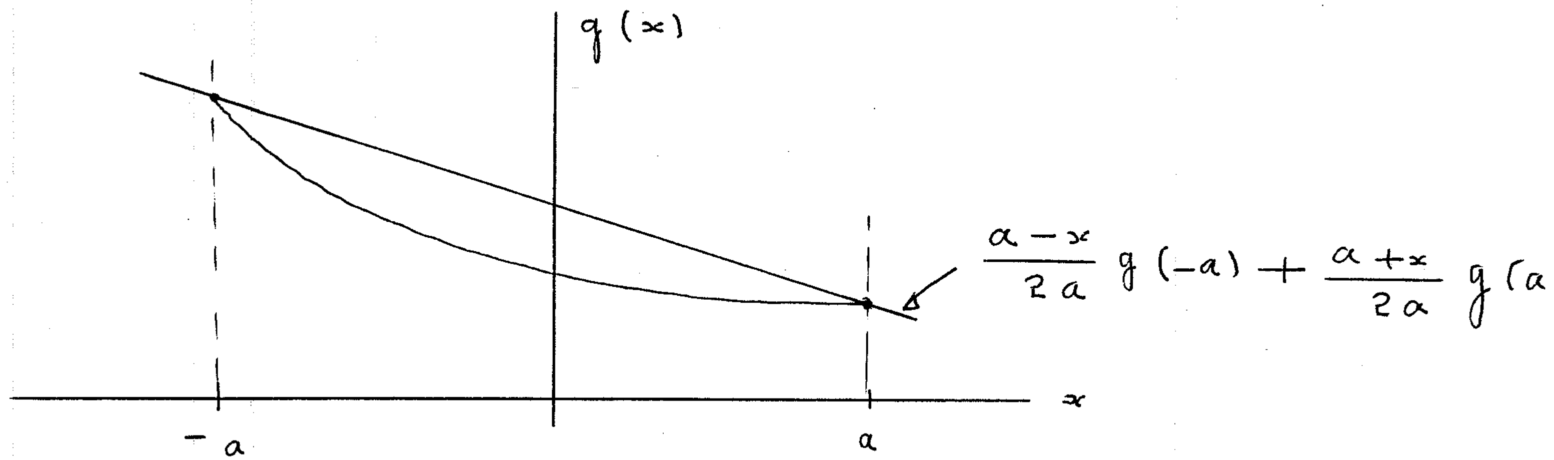
$$< \sum_{k=0}^{\infty} (1-a)^k < \infty. \quad \square$$

1.

2.8.5

(a) Fix $\lambda \in \mathbb{R}$. Put

$$g(x) \triangleq e^{\lambda x} \qquad x \in [-a, a] \quad (1)$$



Since $g(\cdot)$ is convex

$$g(x) \leq \frac{a-x}{2a} g(-a) + \frac{a+x}{2a} g(a), \quad -a \leq x \leq a \quad (2)$$

Then

$$E\{e^{\lambda Y}\} = E g(Y)$$

$$\overset{(2)}{\leq} E\left\{ \frac{a-Y}{2a} \cdot g(-a) + \frac{a+Y}{2a} g(a) \right\}$$

$$\leq \frac{1}{2} g(-a) - \frac{1}{2a} g(-a) E Y + \frac{1}{2} g(a) + \frac{1}{2a} g(a) E Y$$

2

ie $$E e^{\lambda Y} \underset{\underset{EY=0}{\uparrow}}{\leq} \frac{g(a) + g(-a)}{2} = \frac{e^{\lambda a} + e^{-\lambda a}}{2} \quad \text{—— ③}$$

2.8.5

(b) Now $\{Y_k, \; k = 1, 2, \dots\}$ are indep. rvs on $(\Omega, \mathcal{F}, P)$ such that

$$E Y_k = 0 \qquad |Y_k| \leq a \quad \text{—— ④}$$

Put

$$X_n \triangleq \sum_{k=1}^{n} Y_k, \qquad \mathcal{F}_n \triangleq \sigma\{Y_1, Y_2, \dots Y_k\} \quad \text{—— ⑤}$$

Then

$$E[X_n \mid \mathcal{F}_{n-1}] = E\left[\sum_{k=1}^{n-1} Y_k + Y_n \,\Big|\, \mathcal{F}_{n-1}\right]$$

$$= \underbrace{E\left[\sum_{k=1}^{n-1} Y_k \,\Big|\, \mathcal{F}_{n-1}\right]}_{\sum_{k=1}^{n-1} Y_k \,=\, X_{n-1}} + \underbrace{E[Y_n \mid \mathcal{F}_{n-1}]}_{E(Y_n) = 0}$$

since $Y_n \perp\!\!\!\perp$
$Y_n \perp\!\!\!\perp \mathcal{F}_{n-1}$

ie
$$E\left[X_n \mid \mathcal{F}_{n-1}\right] = X_{n-1} \quad \text{——} \ (6)$$

$\boxed{2.8.5}$

Fix $x \in [0, \infty)$ and $\lambda \in (0, \infty)$. Then

$$\sup_{1 \le k \le n} X_k \ge x \iff \exp\left\{\lambda \sup_{1 \le k \le n} X_k\right\} \ge e^{\lambda x}$$

($x \to e^{\lambda x}$ is increasing).

$$\iff \sup_{1 \le k \le n} e^{\lambda X_k} \ge e^{\lambda x}$$

Thus

$$P\left\{\sup_{1 \le k \le n} X_k \ge x\right\} = P\left\{\sup_{1 \le k \le n} Z_k \ge e^{\lambda x}\right\} \quad \text{——} \ (7)$$

where
$$Z_k \doteq e^{\lambda X_k} \quad \text{——} \ (8)$$

From (6) (8) and Thm 2.2.10

$\{(Z_n, \mathcal{F}_n)\}$ is a nonnegative submart —— (9)

and from (9) and Thm 2.4.1

$$P\left\{\sup_{1\le k\le n} Z_k \geq e^{\lambda x}\right\} \leq e^{-\lambda x}\, E[Z_n] \quad \text{——— (10)}$$

2.8.5

Now

$$E[Z_n] = E\left[\exp\left\{\lambda \sum_{k=1}^{n} Y_k\right\}\right]$$

$$= E\left[\prod_{k=1}^{n} e^{\lambda Y_k}\right]$$

$$= \prod_{k=1}^{n} E\left[e^{\lambda Y_k}\right] \quad \text{——— (11)}$$

(since the $Y_1 \ldots Y_n$ are indep.) From (a)

$$E\, e^{\lambda Y_k} \leq \frac{e^{\lambda a} + e^{-\lambda a}}{2} \leq \exp\left\{\frac{\lambda^2 a^2}{2}\right\} \quad \text{——— (12)}$$

combine (12) (11) (10)

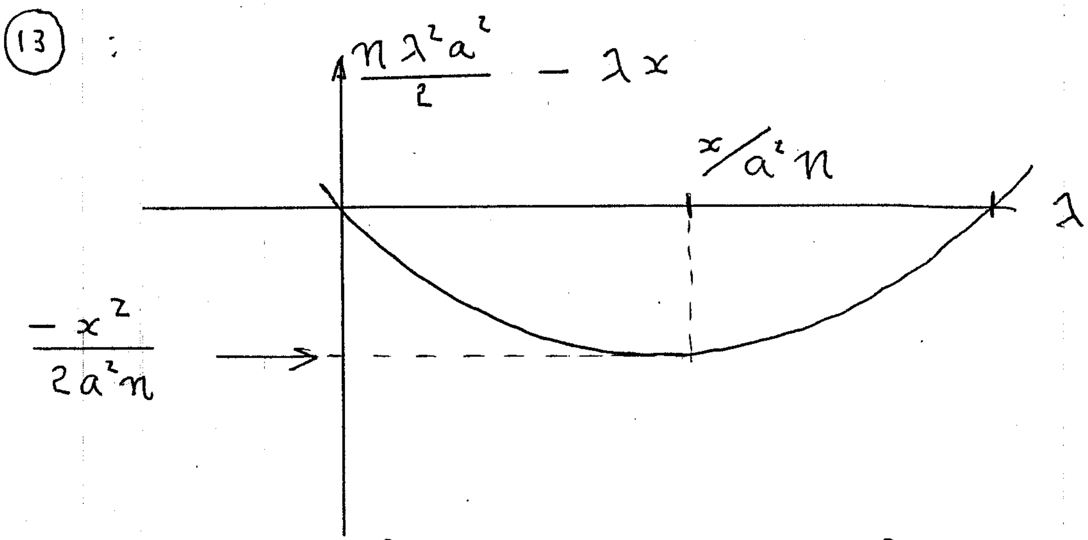
$$P\left\{\sup_{1\le k\le n} Z_k \geq e^{\lambda x}\right\} \leq e^{-\lambda x} \cdot \exp\left\{\frac{n\lambda^2 a^2}{2}\right\}$$

i.e.

$$P\left\{ \sup_{1 \le k \le n} Z_k \right\} \le \exp\left\{ \frac{n\lambda^2 a^2}{2} - \lambda x \right\} \quad \text{—— (13)}$$

Now (13) holds <u>for all</u> $\lambda > 0$. Thus choose $\lambda > 0$ to minimize right side of (13):

2.8.5



$$P\left\{ \sup_{1 \le k \le n} Z_k \ge x \right\} \le \exp\left\{ -\frac{x^2}{2a^2 n} \right\} \quad \text{—— (14)}$$

Result follows from (14) and (7).

2.8.6

(a) observe

$$\{T \geq n\} = \{T < n\}^c$$
$$= \{T \leq n-1\}^c \in \mathcal{F}_{n-1} \quad \text{— (1)}$$

Since $\{X_n\}$ is an indep. sequence

$$X_n \perp\!\!\!\perp \sigma\{X_1 \ldots X_{n-1}\} = \mathcal{F}_{n-1} \quad \text{— (2)}$$

Then

$$E[|X_n| I\{T \geq n\}]$$
$$= E[E(|X_n| I\{T \geq n\} | \mathcal{F}_{n-1})]$$
$$\overset{(1)}{=} E[E[|X_n| \,|\, \mathcal{F}_{n-1}] I\{T \geq n\}] \quad \text{— (3)}$$

and

$$E[|X_n| \,|\, \mathcal{F}_{n-1}] = E[|X_n|] \quad \text{— (4)}$$

(since it follows from (2) that $|X_n| \perp\!\!\!\perp \mathcal{F}_{n-1}$).

2.8.6

then

$$E\left[|X_n|\, I\{T \geq n\}\right] \underset{\text{③ ④}}{=} E\left\{E[|X_n|]\, I\{T \geq n\}\right\}$$

$$= E[|X_n|]\, E\{I\{T \geq n\}\}$$

$$= E[|X_1|]\, P[T \geq n] \quad \text{—— ⑤}$$

Repeating the preceding argument with X_n in place of $|X_n|$ gives

$$E\left[X_n\, I\{T \geq n\}\right] = E[X_1]\, P[T \geq n]$$

$$= \mu\, P[T \geq n] \quad \text{—— ⑥}$$

(b) since $E[T] < \infty$ we have

$$T < \infty \quad \text{a.s.} \quad \text{—— ⑦}$$

3

2.8.6

Then
$$T \overset{\text{trivial!}}{=} \sum_{i=1}^{T} 1 \overset{\text{trivial!}}{=} \sum_{1 \leq i < \infty} I\{T \geq i\} \quad \text{—— (8)}$$

i.e.
$$E[T] \overset{(8)}{=} E\left[\sum_{1 \leq i < \infty} I\{T \geq i\}\right]$$

$$\underset{\text{monotone conv. thm!}}{=} \sum_{1 \leq i < \infty} \underbrace{E[I\{T \geq i\}]}_{P[T \geq i]} = \sum_{1 \leq i < \infty} P[T \geq i] \quad \text{—— (9)}$$

Now
$$S_n \doteq \sum_{i=1}^{n} X_i \quad \text{—— (10)}$$

Then
$$S_T \overset{(10)}{=} \sum_{i=1}^{T} X_i \overset{\text{trivial}}{=} \underbrace{\sum_{1 \leq i < \infty} X_i I\{T \geq i\}}_{\text{sum always makes sense because of (7)}}$$

i.e.

$$S_T = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} X_i \, I\{T \geq i\}}_{\text{limit exists in view of } (7)} \quad \text{——} \; (11)$$

Now suppose we knew that

$$E\left[\lim_{n\to\infty} \sum_{i=1}^{n} X_i \, I\{T \geq i\}\right]$$

$$= \lim_{n\to\infty} E\left[\sum_{i=1}^{n} X_i \, I\{T \geq i\}\right] \quad \text{——} \; (12)$$

Then we have

$$E[S_T] \overset{(11)}{=} E\left[\lim_{n} \sum_{i=1}^{n} X_i \, I\{T \geq i\}\right]$$

$$\overset{(12)}{=} \lim_{n} E\left[\sum_{i=1}^{n} X_i \, I\{T \geq i\}\right]$$

(2.8.6)

$$= \lim_{n \to \infty} \sum_{i=1}^{n} E\left[X_i \, I\{T \geq i\}\right]$$

finitely many summands for each n

$$\overset{⑥}{=} \lim_{n \to \infty} \sum_{i=1}^{n} \mu \, P[T \geq i]$$

$$= \mu \sum_{1 \leq i < \infty} P[T \geq i] \overset{⑨}{=} \mu \, E[T] \quad \text{—— (13)}$$

as required !

So it remains to justify (12). For this we will use LDC.

Put

$$Z \triangleq \underbrace{\sum_{1 \leqslant i < \infty} |X_i| \, I\{T \geq i\}}_{\substack{\text{sum makes sense} \\ \text{since summands are } \geq 0 \text{ !}}} \quad \text{—— (15)}$$

2.8.6

and
$$Z_n \triangleq \sum_{i=1}^{n} X_i \, I\{T \geq i\} \quad \text{—— (16)}$$

Then of course
$$|Z_n| \leq Z \quad \text{—— (17)}$$

Moreover
$$E[Z] \overset{(15)}{=} E\left\{ \sum_{1 \leq i < \infty} |X_i| \, I\{T \geq i\} \right\}$$
$$= \sum_{1 \leq i < \infty} E\left[|X_i| \, I\{T \geq i\} \right]$$

monotone conv. thm!

$$\overset{(5)}{=} \sum_{1 \leq i < \infty} E|X_1| \, P[T \geq i]$$
$$= E|X_1| \sum_{1 \leq i < \infty} P[T \geq i] \overset{(9)}{=} E|X_1| \cdot ET \overset{\text{why?}}{<} \infty \quad \text{(18)}$$

Then

$$E\left[\lim_{n\to\infty}\sum_{i=1}^{n} X_i\, I\{T \geq i\}\right]$$

$$\overset{16}{=} E\left[\lim_{n\to\infty} Z_n\right]$$

$$= \lim_{n} E[Z_n] \overset{16}{=} \lim_{n\to\infty} E\left(\sum_{i=1}^{n} X_i\, I\{T\geq i\}\right)$$

from 18 17 and LDC

which is 12.

Since $A \in \mathcal{F}_T$ it follows that

(5.1) $$A_k \triangleq A \cap \{T=k\} \in \mathcal{F}_k \quad (2.8.7)$$

for all $k = 0, 1, 2 \ldots$ thus

(5.2) $$A_k \cap \{T=k\} = A \cap \{T=k\} \in \mathcal{F}_k$$

$\forall \; k = 0, 1, 2 \ldots$

Now, for each integer $k = 0, 1, 2 \ldots$ we have

(5.3) $$\mathbb{P}\left(A \cap \{T=k\} \cap \{X_{T+n} \in I_n, \; \forall n = 1, 2 \ldots m\}\right)$$

$$= \mathbb{P}\left(A \cap \{T=k\} \cap \{X_{k+n} \in I_n, \; \forall n = 1, 2 \ldots m\}\right)$$

$$= \mathbb{P}\left(A_k \cap \{T=k\} \cap \{X_{k+n} \in I_n, \; \forall n = 1, 2 \ldots m\}\right)$$

Since $\{X_n, \; n = 0, 1, 2 \ldots\}$ is an indep. sequence we see that $A_k \cap \{T=k\} \in \mathcal{F}_k$ and $\{X_{k+n} \in I_n, \; \forall n = 1, 2 \ldots m\}$ are indep.

events, thus, from (5.3):

2.8.7

$$(5.4)\quad \underline{P}\left(A \cap \{T = k\} \cap \{X_{T+n} \in I_n,\ \forall\, n = 1,2\ldots m\}\right)$$

$$= \underline{P}(A_k \cap \{T = k\}) \cdot \underline{P}(X_{k+n} \in I_n,\ \forall\, n = 1,2\ldots m)$$

$$= \underline{P}(A \cap \{T = k\}) \prod_{n=1}^{m} \mu(I_n)$$

where second equality follows by (3.2) & independence of $\{X_{k+n},\ n = 1,2\ldots m\}$.

Summing (5.4) over all $k = 0,1,2\ldots$

and using

$$\bigcup_{0 \le k < \infty} \{T = k\} = \{T < \infty\} = \Omega$$

gives desired result.

Clearly T is the debut of $(-\infty, 0]$ by the adapted sequence $(X_n, \mathcal{F}_n)$, thus (see Example 2.1.6) one sees that T is a $\{\mathcal{F}_n\}$ - stopping time. Moreover

$$T \wedge n \leq n \; ;$$

thus since $\{(X_n, \mathcal{F}_n),\ n = 0, 1, \ldots\}$ is a supermart. we see from the optional sampling thm. for <u>bounded</u> stopping times that

$$E\left[X_n \mid \mathcal{F}_{T \wedge n}\right] \leq X_{T \wedge n} \quad \text{a.s.}$$

$$\forall\, n = 0, 1, 2 \ldots$$

hence, since $\{T \leq n\} \in \mathcal{F}_{T \wedge n}$:

$$0 \leq E\left[X_n \,;\, T \leq n\right] \leq \underbrace{E\left[X_{T \wedge n} : T \leq n\right]}_{E\left[X_T \,:\, T \leq n\right] = 0}$$

2.8.8

and

$$E[X_T : T \leq n] = 0$$

(by defn. of T). Thus

$$E[X_n I\{T \leq n\}] = 0 , \quad \forall n = 0,1,2\ldots$$

or

$$X_n . I\{T \leq n\} = 0 \quad a.s \quad \_\_\_ (1)$$

$$\forall n = 0, 1, 2 \ldots$$

Now put

$$A \triangleq \{\omega : X_n I\{T \leq n\} = 0, \quad \forall n = 0, 1, 2, \ldots\} \_\_ (2)$$

From (1) and (2):

$$\underline{P} (A) = 1 \quad \_\_\_\_ (3)$$

Now

$$A \cap \{T < \infty\} \subset \{T < \infty\}$$

2.8.8

so that (3) gives

$$P[A \cap \{T < \infty\}] = P\{T < \infty\} \quad \text{—— (4)}$$

But, from (2):

$$A \cap \{T < \infty\}$$

$$= \{T < \infty\} \cap \{X_n = 0, \ \forall n \geq T\} \quad \text{—— (5)}$$

thus desired result follows from (4), (5).

2.8.9

[1] We are given

$$|X_n| \leq Z \qquad E[Z] < \infty \quad \text{—— (1)}$$

$$X_n \longrightarrow X \quad \text{a.s.} \quad \text{—— (2)}$$

From (1) (2)

$$|X| \leq Z \quad \text{a.s.} \quad \text{—— (3)}$$

Put

$$Y_m \triangleq \sup_{n \geq m} |X_n - X|, \quad m = 0, 1, 2 \ldots \text{—— (4)}$$

Then

$$|X_n - X| \leq |X_n| + |X| \overset{(1)(3)}{\leq} 2Z \quad \text{—— (5)}$$

i.e.

$$\sup_{n \geq m} |X_n - X| \leq 2Z, \quad \text{all } m = 0, 1, 2 \ldots \text{—— (6)}$$

From (6) (5)

$$0 \leq Y_m \leq 2Z, \quad m = 0, 1, 2 \ldots \text{—— (7)}$$

2.8.9

In particular from 7 1

$$E[|Y_m|] < \infty \quad \text{———} \; (8)$$

Put

$$\Delta_n \triangleq \left| E[X_n | \mathcal{F}_n] - E[X | \mathcal{F}_\infty] \right| \quad \text{———} \; (9)$$

Then

$$\Delta_n = \Big| E[X_n | \mathcal{F}_n] - E[X | \mathcal{F}_n] + E[X | \mathcal{F}_n] - E[X | \mathcal{F}_\infty] \Big|$$

$$\leq \left| E[X_n - X | \mathcal{F}_n] \right| + \left| E[X | \mathcal{F}_n] - \cancel{E}[X | \mathcal{F}_\infty] \right|$$

$$\underset{\substack{\uparrow \\ \text{Jensen}}}{\leq} E\big[|X_n - X| \,\big|\, \mathcal{F}_n\big] + \left| \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \right| \quad \text{———} \; (10)$$

2.8.9

Now fix positive integer m. From (4)

$$Y_m \geq |X_n - X|, \quad \text{all } n \geq m \quad \text{— (11)}$$

thus

$$E[Y_m \mid \mathcal{F}_n] \underset{\text{from (11)}}{\geq} E[|X_n - X| \mid \mathcal{F}_n], \quad \text{all } n \geq m \quad \text{— (12)}$$

From (12) (10)

$$\Delta_n \leq E[Y_m \mid \mathcal{F}_n] + \left| E[X \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_\infty] \right|,$$

$$\text{all } n \geq m \quad \text{— (13)}$$

From the elementary inequality

$$\limsup_n (a_n + b_n) \leq \limsup_n a_n + \limsup_n b_n$$

(which holds for sequences $\{a_n\}$ + $\{b_n\}$ of real numbers)

2.8.9

and (13) we get

$$\limsup_n \Delta_n \leq \limsup_n E[Y_m \mid \mathcal{F}_n] + \limsup_n \left| E[X \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_\infty] \right| \quad \text{— (14)}$$

From (1) (3) have

$$E|X| < \infty \quad \text{— (15)}$$

and from (15) and the Lévy thm 2.7.6

$$\lim_n E[X \mid \mathcal{F}_n] = E[X \mid \mathcal{F}_\infty] \quad \text{a.s.}$$

thus

$$\lim_n \left| E[X \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_\infty] \right| = 0 \quad \text{a.s.}$$

thus (of course!)

$$\limsup_n \left| E[X \mid \mathcal{F}_n] - E[X \mid \mathcal{F}_\infty] \right| = 0 \quad \text{a.s.} \quad \text{— (16)}$$

2.8.9

From (8) and Lévy Thm 2.7.6 (again!)

$$\lim_n E[Y_m \mid \mathcal{F}_n] = E[Y_m \mid \mathcal{F}_\infty]$$

(remember that m is fixed!)

thus

$$\limsup_n E[Y_m \mid \mathcal{F}_n] = E[Y_m \mid \mathcal{F}_\infty] \quad \text{—— (17)}$$

In view of (17) (16) (14)

$$\limsup_n \Delta_n \leq E[Y_m \mid \mathcal{F}_\infty] \quad \text{for each } m = 0, 1, \ldots \quad \text{—— (18)}$$

Next observe that

$$\lim_m Y_m = 0 \quad \text{a.s.} \quad \text{—— (19)}$$

To check (19) observe from (2) that

$$\lim_n |X_n - X| = 0 \quad \text{a.s.}$$

2.8.9

thus (of course!)

$$\underbrace{\limsup_{n \to \infty} |X_n - X|}_{} = 0 \quad \text{a.s.}$$

which is the same as

$$\overbrace{\lim_{m \to \infty} \left[\sup_{n \geq m} |X_n - X| \right]} = 0 \quad \text{a.s.}$$

which is 19.

From 19, 7, $E[Z] < \infty$, and LDC for conditional expectation

$$\lim_{m \to \infty} E[Y_m \mid \mathcal{F}_\infty] = 0 \quad \text{a.s.} \qquad (20)$$

and, from 20 and 18

$$\limsup_{n} \Delta_n \leq 0 \qquad (21)$$

But $\Delta_n \geq 0$ thus from 21

$$\lim_{n} \Delta_n = 0$$

as required.