Ideal periodic sampling of analog signals

- **Uniform sampling** of an analog signal $x_c(t)$ produces
  \[
  x[n] \triangleq x_c(t) \bigg|_{t=nT} = x_c(nT), \quad n \in \mathbb{Z},
  \]
  where $T > 0$ is a **sampling period**.

- The system that implements this operation is known as an **ideal analog-to-digital converter (ADC)** or **ideal sampler**.

- In general, an infinite number of signals can fit the same $x[n]$.

![Graph showing multiple signals](image)

- **Are the samples** $x[n]$ **sufficient to describe uniquely** $x_c(t)$ **and, if so, how can** $x_c(t)$ **be reconstructed from** $x[n]$?
Ideal uniform sampling

- Signal $x_c(t)$ and its *continuous-time Fourier transform* (CTFT) $X_c(j\Omega)$ are uniquely related as
  \[ X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt, \quad x_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega) e^{j\Omega t} d\Omega. \]

- At the same time, the sequence $x[n]$ and its DTFT $X(e^{j\omega})$ are uniquely related as
  \[ X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}, \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega. \]

- Above, the physical frequency $\Omega$ and the normalized circular frequency $\omega$ are related according to
  \[ \omega = \Omega T. \]
To find a connection between $X(e^{j\omega})$ and $X_c(j\Omega)$, we introduce an *impulse train* signal

$$p_T(t) = \sum_n \delta(t - nT),$$

with

$$P_T(j\Omega) = \frac{2\pi}{T} \sum_k \delta\left(j(\Omega - \frac{2\pi}{T}k)\right).$$
Now, define $x_T(t) = x_c(t) \cdot p_T(t) = \sum_n x[n] \delta(t - nT)$.

The CTFT of $x_T(t)$ can be computed in two different ways.

First:

$$
\int_{-\infty}^{\infty} \left[ \sum_n x[n] \delta(t - Tn) \right] e^{-j\Omega t} dt = \sum_n x[n] e^{-j\Omega T t} = X(e^{j\Omega T}).
$$

Second:

$$
\frac{1}{2\pi} (P_T * X_c)(j\Omega) = \frac{1}{T} \sum_k X_c \left( j \left( \Omega - \frac{2\pi k}{T} \right) \right).
$$

Equating both results and using $\omega = \Omega T$, we conclude that

$$
X(e^{j\omega}) = \frac{1}{T} \sum_k X_c \left( j \frac{\omega - 2\pi k}{T} \right).
$$
Hence, the DTFT of $x[n]$ can be obtained by first periodizing $X_c(j\Omega)$ around all integer multiples of $\Omega_s$, followed by rescaling $\Omega = \omega/T$. 

\[
X(e^{j\omega}) = \frac{1}{T} \sum_{k} X_c(j(\omega - 2\pi k)/T)
\]
Aliasing

(a)

(b)

(c)
Two conditions obviously are necessary to avoid *aliasing* (i.e., overlapping of the spectral “replicas”), namely

\[ X_c(j\Omega) = 0, \quad |\Omega| > \Omega_H \quad \text{and} \quad \Omega_s > 2\Omega_H. \]

We will use the following terminology:

The minimum sampling frequency required to avoid aliasing is $2F_H$, which is called the *Nyquist rate*. 
Sampling theorem (cont.)

- If $x_c(t)$ is band-limited and $\Omega_s > 2\Omega_H$, we have

$$X_c(j\Omega) = \begin{cases} 
TX(e^{j\Omega T}), & |\Omega| \leq \Omega_s/2, \\
0, & |\Omega| > \Omega_s/2,
\end{cases}$$

which can be used to recover $x_c(t)$ via inverse FT.

**Sampling Theorem (Shannon, 1949)**

Let $x_c(t)$ be a continuous-time band-limited signal with $X_c(j\Omega) = 0$, for $|\Omega| > \Omega_H$. Then $x_c(t)$ can be uniquely determined by its samples $x[n] = x_c(nT)$, $n \in \mathbb{Z}$, if the sampling frequency $\Omega_s = 2\pi F_s$ satisfies

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_H,$$

or, equivalently,

$$F_s = \frac{1}{T} \geq 2F_H.$$
The process of fitting a continuous function to a set of samples is known as an interpolation.

A general formula that describes a broad class of reconstruction processes is given by

$$x_r(t) = \sum_n x[n] g_r(t - nT),$$

where $g_r(t)$ is an interpolating/reconstruction function/kernel.

Thus, if $\text{supp}(g) \geq T$, the addition of the overlapping copies “fills the gaps” between samples.
In the Fourier domain, the interpolation formula becomes

\[ X_r(j\Omega) = \sum_n x[n] G_r(j\Omega) e^{-j\Omega T n} = G_r(j\Omega) \sum_n x[n] e^{-j\Omega T n}, \]

which suggests

\[ X_r(j\Omega) = G_r(j\Omega) X(e^{j\Omega T}) \]

Specifically, if we choose \( g_r(t) \) so that

\[ G_r(j\Omega) \triangleq G_{BL}(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_s/2, \\ 0, & |\Omega| > \Omega_s/2, \end{cases} \]

then \( X_r(j\Omega) = X_c(j\Omega) \) and, therefore, \( x_r(t) = x_c(t) \).
Evaluating the inverse Fourier transform of $G_{BL}(j\Omega)$, we obtain

$$g_r(t) \triangleq g_{BL}(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}(t/T),$$

in which case we obtain:

The ideal interpolation formula

$$x_c(t) = \sum_n x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

The system used to implement the ideal interpolation is known as an *ideal DAC*.
To understand the meaning the ideal interpolation we look more closely at the sinc function $g_{BL}(t)$.

Note that $g_{BL}(nT) = \delta[n]$. Thus, it is always true that $x_r(nT) = x_c(nT)$, regardless of whether aliasing occurred during sampling.
The effect of undersampling: aliasing

- If there is no aliasing, then $x_c(t)$ can be recovered by means of inverse CTFT of the central “replica” of $x[n]$.

- If these is aliasing, the spectrum of $x_c(t)$ is no longer recoverable from the spectrum of $x[n]$.

- The “talk-through” system (shown below) is often used to verify the correct operation and limitations of A/D and D/A converters.

We will use this system to analyze the sampling of

$$x_c(t) = \cos(2\pi F_0 t) = \frac{1}{2} e^{j2\pi F_0 t} + \frac{1}{2} e^{-j2\pi F_0 t},$$

for different choices of $F_0$. 
The effect of undersampling: aliasing (cont.)

(a) Spectrum of $x_c(t)$

(b) Spectrum of $x[n]$

(c) Spectrum of $x_1(t)$

$F_0 < \frac{F_s}{2}$

No aliasing
The effect of undersampling: aliasing (cont.)
Let us consider the following representative example, with \( A > 0 \).

\[
x_c(t) = e^{-A|t|} \quad \Rightarrow \quad X_c(j\Omega) = \frac{2A}{A^2 + \Omega^2}.
\]

Sampling \( x_c(t) \) at \( F_s = 1/T \) yields

\[
x[n] = e^{-A|nT|} = (e^{-AT})^{|n|} = a^{|n|}, \text{ where } a \doteq e^{-AT}.
\]

The DTFT of \( x[n] \) is given by

\[
X(e^{j\omega}) = \frac{1 - a^2}{1 - 2a \cos(\omega) + a^2}, \quad \omega = \Omega/F_s
\]

which is \( 2\pi \)-periodic.

The reconstructed signal \( y_r(t) \) (with \( Y_r(j\Omega) = X(e^{j\Omega T})G_{BL}(j\Omega) \)) will exhibit substantial artifacts due to aliasing as shown next.
Aliasing in non-bandlimited signals (cont.)
In many applications it is advantageous to filter a continuous-time signal using a discrete-time filter using:

\[ x[n] = x_c(t) \Big|_{t=nT} = x_c(nT) \]

Discrete-Time LTI System:
The operation of a discrete-time LTI system is described by

\[ y[n] = (x \ast h)[n] \quad \text{or} \quad Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}). \]

The input to the ideal DAC is a sequence of numbers \( y[n] \) and the output is a continuous-time signal \( y_r(t) \) obtained using:

**Ideal DAC**

\[ y_r(t) = \sum_n y[n] g_{BL}(t - nT), \quad Y_r(j\Omega) = G_{BL}(j\Omega) Y(e^{j\Omega T}). \]

Note that \( y[n] \) is just an indexed sequence of numbers; the physical time information is provided by the user through \( G_{BL}(j\Omega) \).
Discrete processing of analog signals (cont.)

Spectrum scaling and periodization

\[ X(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_e \left( \Omega - \frac{2\pi}{T} k \right) \]

Frequency-scale normalization \( \Omega = \frac{\omega}{T} \)  

Time-scale normalization \( n = \frac{t}{T} \)

Convolution

\[ y[n] = h[n] \ast x[n] \]

Multiplication

\[ Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}) \]

Frequency-scale unnormalization \( \Omega = \omega / T = \omega F_s \)

Time-scale unnormalization \( t = nT \)

Ideal DAC

\[ y_e(nT) = y[n] \]

Spectrum scaling and unperiodization

\[ Y_e(j\Omega) = G_r(j\Omega) Y_e(e^{j\Omega T}) \]
The overall system from the previous slide can be described as

\[ Y_r(j\Omega) = G_{BL}(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}) = 
\]

\[ = G_{BL}(j\Omega)H(e^{j\Omega T}) \frac{1}{T} \sum_k X_c \left( j(\Omega - \frac{2\pi k}{T}) \right). \]

If there is no aliasing, i.e., \( X_c(j\Omega) = 0 \) for \( |\Omega| > 2\pi(2F_H) \), then

\[ Y_r(j\Omega) = \begin{cases} 
H(e^{j\Omega T})X_c(j\Omega), & \text{if } |\Omega| \leq \pi/T \\
0, & \text{if } |\Omega| > \pi/T 
\end{cases} = H_{\text{eff}}(j\Omega)X_c(j\Omega), \]

where \( H_{\text{eff}}(j\Omega) \) is an effective filter defined as

\[ H_{\text{eff}}(j\Omega) = \begin{cases} 
H(e^{j\Omega T}), & \text{if } |\Omega| \leq \pi/T \\
0, & \text{if } |\Omega| > \pi/T 
\end{cases}. \]
Examples

- **Ideal low-pass filter**: Given $\Omega_c = \omega_c T$, with $0 < \omega_c < \pi$,

$$H_{\text{eff}}(j\Omega) = \begin{cases} 
1, & \text{if } |\Omega| \leq \omega_c / T \\
0, & \text{if } |\Omega| > \omega_c / T.
\end{cases}$$

- **Ideal differentiator**: If $x_c(t)$ is band-limited with $\Omega_H \leq \pi / T$, then

$$H_{\text{eff}}(j\Omega) = \begin{cases} 
j\Omega, & \text{if } |\Omega| \leq \pi / T \\
0, & \text{if } |\Omega| > \pi / T.
\end{cases}$$

Note that, in this case, $H(e^{j\omega}) = H_{\text{eff}}(j\omega / T)$ and $h[n]$ is given by

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{j\omega}{T} e^{j\omega n} d\omega = \frac{1}{T} \begin{cases} 
0, & n = 0 \\
\cos(\pi n) / n, & n \neq 0
\end{cases},$$

which could be interpreted as a “derivative” of a sinc filter.
Let $x_c(t)$ be a real-valued signal that is band-limited to the range $(\Omega_L, \Omega_H)$, viz. $X(j\Omega) = 0$ for $|\Omega| < \Omega_L$ and $|\Omega| > \Omega_H$.

The centre frequency and bandwidth of such a bandpass signal are defined as $\Omega_C = (\Omega_L + \Omega_H)/2$ and $B = (\Omega_H - \Omega_L)$, respectively.

Let us assume there exists an integer $K > 0$ such that $\Omega_H = KB$ and set $\Omega_s = 2B$.

Recall that the spectrum of the sampled signal $x[n] = x_c(nT)$ is defined as

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_k X_c(j(\Omega - k\Omega_s)).$$
One can see that, for the chosen parameters, neither of the “replicas” overlaps with the others.
Thus, if we multiply $X(e^{j\Omega T})$ by $G_r(j\Omega)$, we can recover $X_c(j\Omega)$, and hence $x_c(t)$, exactly.

The ideal reconstruction process is given by

$$x_c(t) = \sum_n x_c(nT) g_r(t - nT)$$

where $g_r(t)$ is the modulated ideal band-limited interpolation function given by

$$g_r(t) = 2 \frac{\sin(0.5Bt)}{\pi t} \cos(\Omega_C t).$$

**Conclusion**

A sampling rate of $F_s = 2(F_H - F_L)$ is adequate for aliasing-free sampling of a bandpass signal if $K = F_H/(F_H - F_L)$ is an integer.
Practical sampling and reconstruction differ from ideal sampling and reconstruction in three fundamental aspects:

1. Practical analog signals are rarely band-limited.
2. $x[n]$ has to be quantized (Section 11).
3. The ideal DAC is practically unrealizable.

A more realistic model of a practical system for digital processing of continuous-time signals is shown below.
The *antialiasing filter* band-limits the input signal to the folding frequency.

Note that practical signals are noisy and hence not band-limited.

The conversion process can be adversely affected if the voltage is changing during the conversion time.

To avoid this problem, it is standard to use a *sample-and-hold circuit*.

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![Sample-and-Hold Diagram](image-url)
The role of S/H is to sample $x_c(t)$ as instantaneously as possible and to hold the sample value as constant as possible til the next sample.

So, the output of the S/H circuit can be modelled as a staircase (or, more formally, piecewise constant) waveform.

Note that the S/H system is linear but time-varying.
A band-limited signal can be reconstructed as

$$x_r(t) = \sum_n x[n] g_{BL}(t - nT) = \sum_n x[n] \text{sinc}(t/T - n).$$

Replacing sinc by an arbitrary function $g_r(t)$ can be used to build a \textit{practical digital-to-analog converters (DAC)}, with $g_r(t)$ referred to as its \textit{characteristic pulse}.

At each $t = nT$, the converter generates a pulse $g_r(t - nT)$ scaled by $x[n]$, thus producing

$$x_r(t) = \sum_n x[n] g_r(t - nT),$$

where (as apposed to sinc) $g_r(t)$ is usually chosen to have a finite support.
In particular, in the case of the switch-and-hold (SH) DAC, the characteristic pulse is chosen to be

\[ g_{SH}(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}, \]

resulting in

\[ x_{SH}(t) = \sum_n x[n] g_{SH}(t - nT). \]

Note that the CTFT of \( g_{SH}(t) \) is given by

\[ G_{SH}(j\Omega) = \frac{\sin(\Omega T/2)}{\Omega/2} e^{-j\Omega T/2}, \]

which is not a band-limited function.

\( G_{SH}(j\Omega) \) is capable of neither eliminating the spectral replicas nor properly normalizing \( X_{SH}(j\Omega) \) in the Nyquist band.
To compensate for the effects of the S/H circuit, we can use an analog post-filtering with

$$H_r(j\Omega) = \begin{cases} \frac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T \\ 0, & \text{otherwise} \end{cases},$$

which satisfies $G_{SH}(j\Omega) H_r(j\Omega) = G_{BL}(j\Omega)$.
Switch-and-hold DAC (cont.)

(a) \[ X(e^{j2\pi FT}) \]

(b) \[ |G_{SH}(j2\pi F)| \]

(c) \[ |X_{SH}(j2\pi F)| = |G_{SH}(j2\pi F) X(e^{j2\pi FT})| \]

(d) \[ |H_s(j2\pi F)| \]

(e) \[ |X_s(j2\pi F)| = |H_s(j2\pi F) X_{SH}(j2\pi F)| \]