Chapter 4

Receiver Techniques for Fading Channels
Outline

- Optimum Receiver for Fading Channels
- Error Rate Performance over Fading Channels
  Examples: BPSK and Differential PSK
- Diversity Techniques for Fading Channels (Space, frequency, time diversity)
- Space Diversity
  - Receive Diversity (Maximal Ratio Combining, Selection Combining, Equal Gain Combining)
  - Transmit Diversity
- Equalization for Frequency-Selective Channels
Optimum Receiver for Fading Channels

- Consider a frequency-flat Rayleigh fading channel, where the complex fading coefficient $\alpha$ is assumed to be constant over one symbol duration

$$r(t) = \alpha s_m(t) + z(t) \Leftrightarrow r = \alpha s_m + z$$

- Recall from Ch. 3, that the ML optimal decoder rule over AWGN is

$$\min_m \|r - s_m\|^2$$

- Assuming that perfect knowledge of fading coefficient is available at the decoder (i.e. coherent receiver), the decision rule is given by

$$\min_m \|r - \alpha s_m\|^2 \Leftrightarrow \min_m \left(\|r\|^2 - 2 \Re(r \cdot \alpha^* s_m^*) + |\alpha|^2 \|s_m\|^2 \right)$$

- Under the assumption of constant-energy modulation set, e.g. M-PSK, the decision rule simplifies to

$$\max_m \Re(r \cdot \alpha^* s_m^*)$$

- In the following two examples, we investigate how the presence of fading affects the error rate performance of some common modulation forms.
Example: Error Probability for BPSK over Fading Channels

\[
\begin{align*}
\text{Decision regions} \\
\hat{b} = 1 & \quad \hat{b} = 0 \\
-\sqrt{E} & \quad \sqrt{E}
\end{align*}
\]

Here we have two possible alternatives, therefore we can use a “threshold detector” as an optimal detector. Decision rule is given by \(\text{Re}(r\alpha^*) \geq 0\)

Let \(P(e|\alpha)\) denote the “conditional” error probability.

Then we can perform an expectation to get the final result \(P(e) = E_{\alpha}[P(e|\alpha)]\)

\[
\begin{align*}
P(e|\alpha) &= P(\hat{b} = 0|b = 1, \alpha)P(b = 1) + P(\hat{b} = 1|b = 0, \alpha)P(b = 0) \\
P(b = 0) &= P(b = 1) = 1/2 \quad \text{Equally probable messages} \\
P(\hat{b} = 0|b = 1, \alpha) &= P(\hat{b} = 1|b = 0, \alpha) \quad \text{Due to symmetry}
\end{align*}
\]
**Example:** Error Probability for BPSK over Fading Ch. (cont’d)

Under the assumption that \( b=1 \) is sent

\[
P\left( \hat{b} = 0 \mid b = 1, \alpha \right) = P\left( \text{Re}[r\alpha^*] > 0 \mid b = 1, \alpha \right)
\]

\[
= P\left( \text{Re}(-\sqrt{E}\alpha + z\alpha^*) > 0 \mid \alpha \right)
\]

\[
= P\left( \text{Re}(z\alpha^*) > \sqrt{E} |\alpha|^2 \mid \alpha \right)
\]

\[
= P\left( v > \sqrt{E} |\alpha|^2 \mid \alpha \right)
\]

\[
r = -\sqrt{E}\alpha + z
\]

\[
v = \text{Re}(z\alpha^*)
\]

What is the distribution of \( v \)?

\( z \) is complex Gaussian distributed, i.e. \( z_R, z_I \sim N(0, N_0/2) \).

\[
E[v] = 0 \quad E[v^2] = \frac{N_0}{2} |\alpha|^2 = \frac{N_0}{2} a^2
\]

Recall that the fading coefficient is given as \( \alpha = a \exp(j\varphi) \) \quad |\alpha|^2 = a^2

\[
v = \text{Re}(z\alpha^*) \sim N\left(0, \frac{N_0}{2} a^2\right) \quad \rightarrow \quad f(v) = \frac{1}{\sqrt{\pi N_0 a^2}} \exp\left(-\frac{v^2}{N_0 a^2}\right)
\]
Example: Error Probability for BPSK over Fading Ch. (cont’d)

\[ P(\hat{b} = 0|b = 1, \alpha) = P(v > \sqrt{E}|\alpha|^2|\alpha) \]

\[ = \frac{1}{\sqrt{\pi N_0 a^2 \sqrt{2Ea^2}}} \int_{-\infty}^{\infty} \exp\left(-\frac{v^2}{N_0 a^2}\right) dv \]

\[ = \frac{1}{\sqrt{2\pi \sqrt{2Ea^2/N_0}}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \]

\[ = Q\left(\sqrt{\frac{2E}{N_0}} a^2\right) \quad \text{Compare with the expression in Ch.3 derived for AWGN.} \]

where the \( Q \)-function is defined as \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2} dy \)

➢ So far, we have worked on the “conditional” probability, i.e., we treated the channel magnitude, \( a \), as a constant term

\[ P(e|\alpha) = P(\hat{b} = 0|b = 1, \alpha) = Q\left(\sqrt{\frac{2E}{N_0}} a^2\right) \quad \rightarrow \quad P(e) = E[P(e|a)] \]
**Example:** Error Probability for BPSK over Fading Ch. (cont’ d)

\[
P(e) = E_P[e|a] = \int P(e|a)f(a)da
\]

\[
= \int 2a \exp(-a^2)Q \left( \frac{2a^2E}{N_0} \right) da
\]

To compute the above integral, we use “integration by parts”, i.e. \( \intudy = uy - \int ydu \)

\[
dy = 2a \exp(-a^2)da \Rightarrow y = -\exp(-a^2) \quad u = Q \left( a \frac{2E}{N_0} \right) \Rightarrow du = ?
\]

Here, we use *Leibnitz Rule* to compute \( du \), i.e. derivative of \( Q \) function

\[
g(x) = \int_{a(x)}^{b(x)} f(x,t) dt \quad \frac{dg}{dx} = \frac{db}{dx} f(x,b(x)) - \frac{da}{dx} f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x} dt
\]

\[
Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{t^2}{2} \right) dt \quad \frac{dQ}{dx} = -\frac{1}{\sqrt{2\pi}} \exp\left( -\frac{x^2}{2} \right)
\]

Note that upper limit and the integrand are independent of \( x \).
Example: Error Probability for BPSK over Fading Ch. (cont’d)

- Using the result from the previous page, we have

\[
u = Q \left( a \sqrt{\frac{2E}{N_0}} \right) \Rightarrow du = -\sqrt{\frac{E}{\pi N_0}} \exp \left( -a^2 \frac{E}{N_0} \right)
\]

- Once we identify and compute \( u, y, du, dy \), we use \( \int u dy = uy - \int y du \) to yield

\[
P(e) = -\exp \left( -a^2 \right) Q \left( a \sqrt{\frac{2E}{N_0}} \right)_{0}^{\infty} - \sqrt{\frac{E}{\pi N_0}} \int_{0}^{\infty} \exp \left[ -a^2 \left( 1 + \frac{E}{N_0} \right) \right] da
\]

\[= 1/2\]

\[
\int_{0}^{\infty} \exp \left[ -a^2 \left( 1 + \frac{E}{N_0} \right) \right] da \quad \text{Variable change} \quad s = \sqrt{2a} \sqrt{1 + E/N_0}
\]

\[
= \frac{1}{\sqrt{2}} \int_{\frac{1}{\sqrt{1 + E/N_0}}}^{\infty} \exp \left( -\frac{s^2}{2} \right) ds = \frac{1}{\sqrt{2}} \sqrt{\frac{2\pi}{1 + E/N_0}} \int_{0}^{\infty} \exp \left( -\frac{s^2}{2} \right) ds
\]

\[Q(0) = 1/2\]
Example: Error Probability for BPSK over Fading Ch. (cont’d)

Using the integration results from the previous page, we have

\[
P(e) = -\exp(-a^2)Q\left(a\sqrt{\frac{2E}{N_0}}\right) - \sqrt{\frac{E}{\pi N_0}} \int_0^\infty \exp\left[-a^2\left(1 + \frac{E}{N_0}\right)\right] da
\]

\[
= 1/2
\]

Expanding into Taylor Series and keeping the first two terms, i.e.

\[
f(x) = f(a) + \frac{1}{1!} \frac{df(x)}{dx} \bigg|_{x=a} (x-a) + \frac{1}{2!} \frac{d^2f(x)}{dx^2} \bigg|_{x=a} (x-a)^2 + \ldots
\]

\[
P(e) \approx \frac{1}{4 \frac{E}{N_0}}
\]

\[\Rightarrow\]

- \(P(e)\) decreases **linearly** with SNR over Rayleigh fading channel!
- Contrast this with its exponential decrease over AWGN channel.
To achieve a BER of $P_e = 10^{-3}$

- For AWGN, SNR=6.8dB
- For a Rayleigh fading channel, SNR=24dB

Example: Error Probability for BPSK over Fading Ch. (cont’d)
Effect of Time-Selectivity

We consider a time-selective fading channel where the fading changes symbol-to-symbol in a correlated manner. Coherent reception is not suitable for such environments as the phase tracking becomes very difficult, even intractable. Here, we consider DPSK.

\[
y(t) = y_i(t) + jy_Q(t) = a_k Ah(t) + n(t)
\]

\[
y_k = \int_{kT}^{(k+1)T} y(t) dt = a_k A \int_{kT}^{(k+1)T} h(t) dt + \int_{kT}^{(k+1)T} n(t) dt = a_k AH_k + N_k
\]

\[
H_k = \int_{kT}^{(k+1)T} h(t) dt
\]
Example: Error Probability for DPSK over Fading Channels

\[ y_k = a_k A H_k + N_k \quad N_k \sim (0, 2N_0 T) \quad \text{Ch.3 p.61} \]

\[ H_k \sim \text{complex Gaussian} \]

\[ E(H_k) = \int_{kT}^{(k+1)T} h(t)dt = 0 \quad \text{i.e., Rayleigh fading} \]

\[ \text{Var}(H_k) = E\left[|H_k|^2\right] = \int_{kT}^{(k+1)T} \int_{kT}^{(k+1)T} E\left[ h(t) h^*(u) \right] dt du \approx T^2 \quad \text{(See below)} \]

We assume Clarke’s model, i.e., autocorrelation follows Bessel function

\[ \text{Var}(H_k) = \int_{kT}^{(k+1)T} \int_{kT}^{(k+1)T} R_{HH}(t-u) dt du \]

\[ \approx T \int_{-T}^{T} R_{HH}(\tau) \Lambda\left(\frac{\tau}{T}\right) d\tau \]

\[ \approx TR_{HH}(0) \int_{-T}^{T} \Lambda\left(\frac{\tau}{T}\right) d\tau \]

\[ = T^2 \]

\[ \Lambda(\tau/T) \]

Area = T
Example: Error Probability for DPSK over Fading Ch. (cont’d)

\[ H_{k-1} \overset{\text{def}}{=} \int_{(k-1)T}^{kT} h(t) \, dt \quad H_{k-1} \sim \text{complex Gaussian} \]

\[ E(H_{k-1}) = \int_{(k-1)T}^{kT} E[h(t)] \, dt = 0 \quad \text{Var}(H_k) = E[|H_{k-1}|^2] \approx T^2 \]

Correlation between \( H_k \) and \( H_{k-1} \)

\[ E[H_k H_{k-1}^*] = \int_{kT}^{(k+1)T} \int_{(k-1)T}^{kT} E[h(t) h^*(u)] \, dt \, du \approx \rho T^2 \quad \text{(See below)} \]

\[ \rho = J_0(2\pi f_d T) \]

\[ E[H_k H_{k-1}^*] = \int_{kT}^{(k+1)T} \int_{(k-1)T}^{kT} R_{HH}(t-u) \, dt \, du \quad \text{Approximate the double integration involving Bessel function} \]

\[ \approx T \int_{-T}^{T} R_{HH}(\tau + T) \Lambda \left( \frac{\tau}{T} \right) d\tau \quad \text{Assume Bessel function remains constant over the integration interval} \]

\[ = TR_{HH}(T) \int_{-T}^{T} \Lambda \left( \frac{\tau}{T} \right) d\tau \quad \text{def} \]

\[ \rho = R_{HH}(T) = J_0(2\pi f_d T) \]

\[ = \rho T^2 \quad \text{i.e. correlation between two consecutive samples} \]
**Example:** Error Probability for DPSK over Fading Ch. (cont’d)

\[ z_k = y_k y_{k-1}^* = (a_k A H_k + N_k)(a_{k-1} A H_{k-1}^* + N_{k-1}^*) \]

\[ = b_k A^2 H_k H_{k-1}^* + a_k A H_k N_{k-1}^* + a_{k-1} A H_{k-1}^* N_k + N_k N_{k-1}^* \]

\[ \hat{b}_k = \text{sgn}\left(\text{Re}[b_k A^2 H_k H_{k-1}^* + a_k A H_k N_{k-1}^* + a_{k-1} A H_{k-1}^* N_k + N_k N_{k-1}^*]\right) \]

Differential encoding

\[ a_k = a_{k-1} b_k \]

The \( a_k \) and \( a_{k-1} \) terms in noise do not effect the statistical properties of the noise, so they can be dropped in the following derivation (i.e., replace \( y_k \) with \( u_k \))

\[ P(e|b_k = 1) = P(\hat{b}_k = -1|b_k = 1) = P(\text{Re}[u_k u_{k-1}^*] < 0) \quad \text{where} \quad u_k = A H_k + N_k \]

Instead of finding out the distribution for \( \text{Re}[u_k u_{k-1}^*] \), we will rewrite it in the following

\[ P(e|b_k = 1) = P(\text{Re}[u_k u_{k-1}^*] < 0) = P(|\omega_2| > |\omega_1|) \]

where we define \( \omega_1 = (u_k + u_{k-1})/2 \) and \( \omega_2 = (u_k - u_{k-1})/2 \).
Example: Error Probability for DPSK over Fading Ch. (cont’d)

\[ E[\omega_1] = \frac{1}{2} E[u_k] + \frac{1}{2} E[u_{k-1}] = 0 \]

\[ E[|\omega_1|^2] = \frac{1}{4} E[|u_k|^2] + \frac{1}{4} E[u_k u_{k-1}^*] + \frac{1}{4} E[u_{k-1} u_k^*] + \frac{1}{4} E[|u_{k-1}|^2] = \frac{(AT)^2 (1 + \rho)}{2} + 2N_0T \]

\[ E[\omega_2] = \frac{1}{2} E[u_k] - \frac{1}{2} E[u_{k-1}] = 0 \]

\[ E[|\omega_2|^2] = \frac{1}{4} E[|u_k|^2] - \frac{1}{4} E[u_k u_{k-1}^*] - \frac{1}{4} E[u_{k-1} u_k^*] + \frac{1}{4} E[|u_{k-1}|^2] = \frac{(AT)^2 (1 - \rho)}{2} + 2N_0T \]

\[ E[\omega_1 \omega_2^*] = E \left[ \left( \frac{u_k + u_{k-1}}{2} \right) \left( \frac{u_k - u_{k-1}}{2} \right)^* \right] = \frac{1}{4} \left\{ E[|u_k|^2] - E[|u_{k-1}|^2] \right\} = 0 \rightarrow \text{independent} \]

In the above calculations, we made use of the following:

\[ E[u_k] = 0 \]

\[ E[u_k u_k^*] = E[|u_k|^2] = A^2 E[|H_k|^2] + E[|N_k|^2] = A^2 T^2 + 2N_0T \]

\[ E[u_k u_{k-1}^*] = A^2 E[H_k H_{k-1}^*] + E[N_k N_{k-1}^*] = \rho A^2 T^2 \]
Example: Error Probability for DPSK over Fading Ch. (cont’d)

\[ |\omega_1| : \text{Rayleigh} \quad f_{|\omega_1|}(\omega) = \frac{\omega}{\sigma_1^2} \exp\left(-\frac{\omega^2}{2\sigma_1^2}\right) \quad \omega \geq 0 \quad \sigma_1^2 = \frac{1}{2} E[|\omega_1|^2] \]

\[ |\omega_2| : \text{Rayleigh} \quad f_{|\omega_2|}(\omega) = \frac{\omega}{\sigma_2^2} \exp\left(-\frac{\omega^2}{2\sigma_2^2}\right) \quad \omega \geq 0 \quad \sigma_2^2 = \frac{1}{2} E[|\omega_2|^2] \]

\[ P(e|b_k = 1) = P(\text{Re}[u_k u_{k-1}^*] < 0) = P(|\omega_2| > |\omega_1|) = E_{\omega}\left[P(|\omega_2| > |\omega_1| = \omega)\right] \]

\[ = \int P(|\omega_2| > \omega | |\omega_1| = \omega) f_{|\omega_1|}(\omega) d\omega \quad \text{and } |\omega_1| \text{ and } |\omega_2| \text{ independent} \]

\[ = \int P(|\omega_2| > \omega) f_{|\omega_1|}(\omega) d\omega \]

\[ P(|\omega_2| > \omega) = \int_0^\infty \frac{\omega}{\omega \sigma_2^2} \exp\left(-\frac{\omega^2}{2\sigma_2^2}\right) d\omega \quad \text{def } u = \frac{\omega}{2\sigma_2^2} \]

\[ = \int_{\omega^2/2\sigma_2^2}^{\infty} \exp(-u) du \]

\[ = \exp\left(-\frac{\omega^2}{2\sigma_2^2}\right) \]
Example: Error Probability for DPSK over Fading Ch. (cont’d)

\[ P(e|b_k = 1) = \int P(\omega_2 > \omega)f_{\omega_1}(\omega)d\omega \]

\[
= \int_0^\infty \frac{\omega}{\sigma_1^2} \exp \left[ -\frac{\omega^2}{2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \right] d\omega \\
= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \int_0^\infty \exp(-\nu) d\nu \\
= \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \frac{1 + (1 - \rho)(E/N_0)}{2 + 2(E/N_0)} \equiv \frac{1}{2(E/N_0)} + \frac{1 - \rho}{2} \\
\]

We consider two extreme cases:

\( \rho \to 1 \) (i.e. slow fading) \( \Rightarrow P(e) \to \frac{1}{2(E/N_0)} \)

\( \frac{E}{N_0} \to \infty \Rightarrow P(e) \to \frac{1 - \rho}{2} \)

There is an “irreducible error floor” induced by the fading channel.
Example: Error Probability for DPSK over Fading Ch. (cont’ d)

The graph illustrates the error probability for Digital Phase Shift Keying (DPSK) over a fading channel. The $\rho$ values (from bottom to top) are 0.7, 0.9, 0.97, 0.99, 0.997, and 1. The x-axis represents the $E_b/N_0$ (Eb/No) in dB, while the y-axis represents the BER (Bit Error Rate). The graph shows how the error probability decreases as $E_b/N_0$ increases, indicating improved performance with higher signal-to-noise ratios. The term "slow fading" refers to a slowly varying signal characteristic, which is relevant in wireless communications for modeling the effects of fading on the transmitted signals.
Example: Error Probability for DPSK over Fading Ch. (cont’d)

We can predict the error floor as follows:

Recall that we use Bessel correlation function, i.e. $\rho = J_0(2\pi f_d T)$

Using Taylor series of the Bessel function

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left( \frac{x}{2} \right)^{2k+n} \quad \Rightarrow \quad J_0(x) \approx 1 - \frac{x^2}{4}$$

The error floor will appear at approximately

$$P(e) = \frac{1 - \rho}{2} = \frac{(\pi f_d T)^2}{2}$$

To ensure an error floor smaller than a certain predetermined $P_{th}(e)$,

$$T < \frac{\sqrt{2P_{th}(e)}}{\pi f_d} \quad \text{or equivalently} \quad R = \frac{1}{T} > \frac{\pi f_d}{\sqrt{2P_{th}(e)}}$$
Diversity Techniques for Fading Channels

- Diversity is the primary technique used to improve performance on a fading channel.

- The main idea behind the diversity is to provide the receiver with multiple versions of the same tx signal over independent channels. It is easy to see that the probability of all signals being faded will be less than the probability that just one is faded!

How to create independent channels needed for diversity?

- **Frequency Diversity:** Use different frequency carriers separated by a distance larger than the coherence bandwidth of the channel. Not bandwidth-efficient!

- **Time Diversity:** Use different time slots separated by an interval longer than the coherence time of the channel. An efficient method to exploit time diversity is to use “channel coding+interleaving” (ECE412)

- **Space Diversity:** Use multiple antennas separated wide enough w.r.t. carrier wavelength
Maximal Ratio Combining (MRC)

\[ r_n(t) = \alpha_n s_i(t) + z_n(t) \iff r_n = \alpha_n s_i + z_n \quad n = 1, 2, \ldots, N \]

- \( N \): Number of RX antennas
- \( \alpha_n \): Complex fading coefficient of the \( n \)th channel

The decision is based on the output of linear combiner

\[ r = \sum_{n=1}^{N} c_n \cdot r_n \quad \text{where} \quad c_n \text{ are the weighting coefficients.} \]
MRC (cont’ d)

We need to determine how we should choose the weighting coefficients. Here, for simplicity, we will consider 1-dimensional signals.

\[
\begin{align*}
\mathbf{r} &= \sum_{n=1}^{N} c_n r_n = \sum_{n=1}^{N} c_n \alpha_n s_i + \sum_{n=1}^{N} c_n z_n \\
&= \text{Signal term} + \text{Noise term}
\end{align*}
\]

Noting that
\[
|s_i|^2 = E \sum_{n=1}^{N} c_n z_n \sim N \left(0, \frac{N_0}{2} \sum_{n=1}^{N} |c_n|^2 \right)
\]

The signal-to-noise ratio at the output of linear combiner is given as

\[
\text{SNR} = \frac{E \left| \sum_{n=1}^{N} c_n \alpha_n \right|^2}{\frac{N_0}{2} \sum_{n=1}^{N} |c_n|^2} \leq \frac{2E \sum_{n=1}^{N} |c_n|^2 \sum_{n=1}^{N} |\alpha_n|^2}{\frac{N_0}{2} \sum_{n=1}^{N} |c_n|^2} = \frac{2E}{N_0} \sum_{n=1}^{N} |\alpha_n|^2
\]

“Schwarz Inequality”
\[
\left| \sum_{n=1}^{N} c_n \alpha_n \right|^2 \leq \sum_{n=1}^{N} |c_n|^2 \sum_{n=1}^{N} |\alpha_n|^2
\]

which holds with equality for \( c_n = \alpha_n^* \)
MRC (cont’ d)

➢ We have already seen that the choice of weighting coefficients as \( c_n = \alpha_n^* \) maximizes SNR. In the following, we will derive the optimal decoder in ML sense and compare it with the linear combiner.

➢ Assume that at the decoder, the perfect knowledge of fading coefficients is available. Based on the observations of the received signals at \( N \) antennas

\[
\mathbf{r} = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N] \quad \text{def} \quad \mathbf{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]
\]

\[
p(\mathbf{r}|\alpha, \mathbf{s}_i) = \prod_{n=1}^{N} p(\mathbf{r}_n|\alpha_n, \mathbf{s}_i) = \prod_{n=1}^{N} \prod_{k=1}^{K} p(r_{n,k}|\alpha_n, s_{i,k}) \quad \text{i.e. } K\text{-dim. vectors}
\]

\[
= \prod_{n=1}^{N} \prod_{k=1}^{K} f(r_{n,k} - \alpha_n s_{i,k})
\]

Under the condition that \( \alpha \) is provided, this follows Gaussian pdf.

\[
= \prod_{n=1}^{N} \prod_{k=1}^{K} \frac{1}{\pi N_0} \exp \left( -\frac{1}{N_0} |r_{n,k} - \alpha_n s_{i,k}|^2 \right)
\]

\[
= \frac{1}{(\pi N_0)^{NK}} \exp \left( -\frac{1}{N_0} \sum_{n=1}^{N} \sum_{k=1}^{K} |r_{n,k} - \alpha_n s_{i,k}|^2 \right)
\]
MRC (cont’ d)

- The ML rule is then given as

\[
\max_i p(r|\alpha, s_i) \Leftrightarrow \min_i \sum_{n=1}^N \sum_{k=1}^K |r_{n,k} - \alpha_n s_{i,k}|^2
\]

\[
\Leftrightarrow \min_i \sum_{n=1}^N \|r_n - \alpha_n s_i\|^2
\]

\[
\Leftrightarrow \min_i \sum_{n=1}^N \left(\|r_n\|^2 - 2 \text{Re}(r_n \cdot \alpha_n^* s_i^*) + |\alpha_n|^2 \|s_i\|^2\right)
\]

\[
\Leftrightarrow \max_i \sum_{n=1}^N \text{Re}(r_n \cdot \alpha_n^* s_i^*)
\]

- This coincides with the decoder structure implemented as “linear combiner with weighting coefficients \( c_n = \alpha_n^* \)”.

Error Probability for MRC-BPSK

\[
P(e) = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{E/N_0}{N + E/N_0}} \sum_{n=0}^{N-1} \left( \frac{2n}{n} \right) \left( \frac{1}{4} + \frac{1}{E} \frac{N}{N_0} \right)^n
\]

Now consider the error probability for various values of \(N\):

\(N = 1\)  \[P(e) = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{E/N_0}{1 + E/N_0}} \sim \frac{1}{4} \frac{E}{E/N_0}\]

Single antenna case, i.e., no diversity

\(N = 2\)  \[P(e) = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{E/N_0}{2 + E/N_0}} \left( 1 + \frac{1}{2 + E/N_0} \right) \sim \frac{3/4}{(E/N_0)^2}\]

Diversity order= Slope of the curve=2

\(N = 3\)  \[P(e) = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{E/N_0}{3 + E/N_0}} \left( 1 + 2 \frac{3/4}{3 + E/N_0} + 6 \left( \frac{3/4}{3 + E/N_0} \right)^2 \right) \sim \frac{135/32}{(E/N_0)^3}\]

Diversity order= Slope of the curve=3
Error Probability for MRC-BPSK (cont’ d)

- In general, it can be shown that asymptotically, i.e. for large SNR,

\[
P(e) \sim \left( \frac{N}{4} \right)^N \left( \frac{2N-1}{N} \right) \left( \frac{E}{N_0} \right)^{-N}
\]

- Diversity order = Slope of the curve = \( N \)

- Error probability decreases inversely with the \( N^{th} \) power of the SNR.

- For the limiting case of \( N \rightarrow \infty \) the performance converges to that of AWGN.
The figure on the previous page assumes “scaled” SNR by receive antenna number to demonstrate the convergence to AWGN case. However, in practical, scaling is not necessary.

Without scaling in figures, the “average” SNR is also increased by a factor of \( N \) over the single antenna link. This improvement is known as “power gain” or “array gain”.
Channel SNR Measurements with Diversity

- Diversity removes deep fades and stabilizes the link.

- $N=1$
- $N=2$
- $N=6$
Selection Combining (SC)

\[ r_n(t) = \alpha_n s_i(t) + z_n(t) \iff r_n = \alpha_n s_i + z_n \quad n = 1,2,..N \]

\( N \) : Number of RX antennas

\( \alpha_n \): Complex fading coefficient of the \( n^{th} \) channel

- The combiner chooses the branch with the highest signal-to-noise ratio.

\[ \max_n \gamma_n = \max_n \left( a_n^2 \frac{E}{N_0} \right) \]

- For BPSK, the “conditional” probability is given as

\[ P(e|a_{\text{max}}) = Q\left( \sqrt{\frac{2E}{N_0}} a_{\text{max}}^2 \right) \]

\( a_{\text{max}} \) Fading coefficient amplitude corresponding to the branch with the highest SNR
Error Probability for SC-BPSK ($N=2$)

$$P(e) = \frac{1}{2} - \left\lfloor \frac{E/N_0}{1 + E/N_0} \right\rfloor + \frac{1}{2} \left\lfloor \frac{E/N_0}{2 + E/N_0} \right\rfloor$$

- The same diversity order (i.e. the same slope) with that of MRC is preserved.
- There is a performance degradation compared to that of MRC (i.e. a horizontal shift)
Equal Gain Combining (EGC)

- Coherent Equal Gain Combining is similar to MRC because the diversity branches are co-phased, but different from MRC because the diversity branches are not weighted.

\[ \alpha_n = a_n e^{j\phi_n} \]

- The decoder has the knowledge of \( \varphi = \{\phi_1, \phi_2, \ldots, \phi_N\} \) and assumes \( a_1 = a_2 = \ldots = a_N = 1 \)

\[ \max_i p(r_i|\varphi, s_i) \Leftrightarrow \min_i \sum_{n=1}^{N} \sum_{k=1}^{K} |r_{n,k} - e^{j\phi_n} s_{i,k}|^2 \Leftrightarrow \min_i \sum_{n=1}^{N} \|r_n - e^{j\phi_n} s_i\|^2 \]

\[ \Leftrightarrow \min_i \sum_{n=1}^{N} \left( \|r_n\|^2 - 2 \text{Re}(r_n \cdot e^{-j\phi_n} s_{i}^*) + |e^{j\phi_n}|^2 \|s_i\|^2 \right) \]

\[ \Leftrightarrow \max_i \sum_{n=1}^{N} \text{Re}(r_n \cdot e^{-j\phi_n} s_{i}) \]

- For the binary case, we can use the following threshold detector

\[ \sum_{n=1}^{N} \text{Re}(r_n e^{-j\phi_n}) \geq 0 \]
Error Probability for EGC-BPSK

\[
P(e) = \frac{1}{2} \left( 1 - \sqrt{1 - \left( \frac{1}{1 + (E/N_0)} \right)^2} \right)
\]

- The same diversity order (i.e., the same slope) with that of MRC is preserved.
- There is a performance degradation compared to that of MRC (i.e. a horizontal shift), but less than that SC suffers from.
Transmit Diversity

- So far, we have studied “space diversity” assuming multiple receive antennas. Another powerful form (though more difficult to implement) to exploit spatial diversity is to use multiple transmit antennas. First, we need to clarify why we need transmit diversity.

- In downlink, receive diversity is difficult to implement as it requires multiple antennas and additional processing at the mobile station. This is not suitable due to size and battery power limitation at mobile ☹️
- Put additional processing and complexity at the base station => TX-Diversity

<table>
<thead>
<tr>
<th></th>
<th>Transmitter</th>
<th>Receiver</th>
<th>Diversity type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uplink</td>
<td>Mobile Station</td>
<td>Base Station</td>
<td>RX-Diversity</td>
</tr>
<tr>
<td>Downlink</td>
<td>Base Station</td>
<td>Mobile Station</td>
<td>TX-Diversity</td>
</tr>
</tbody>
</table>
Transmit Diversity (cont’d)

- Extracting diversity from multiple antennas at the transmitter requires **pre-processing** or **pre-coding** prior to transmission.

- Assume the same signal is transmitted from two different antennas without any processing.

\[
\alpha_1, \alpha_2: \text{Zero mean unit variance complex Gaussian}
\]

\[
r = \sqrt{E} \alpha_1 s + \sqrt{E} \alpha_2 s + z
\]

Now, rewrite the above received signal as

\[
r = \sqrt{E} \left( \frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}} \right) s + z
\]

Defining, \( \alpha = \frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}} \) we have

\[
r = \sqrt{E} \alpha s + z
\]

\[
E[|\alpha|^2] = 1
\]

- We obtain the equivalent single-antenna model. **No diversity is provided!**
Closed-Loop Transmit Diversity  
(channel is known at transmitter)

\[ r = \sqrt{\frac{E}{M}} \sum_{m=1}^{M} w_m \alpha_m s + z \]

Scaling by transmit antenna number \( M \) is necessary to keep the transmit power fixed.

\( \alpha_m \): Fading coefficient from \( m^{th} \) transmit antenna to the receive antenna

\( w_m \): Weighting coefficients

Defining \( \alpha = [\alpha_1, \alpha_2, ..., \alpha_M]^T \) \( w = [w_1, w_2, ..., w_M]^T \), we have \( \sum_{m=1}^{M} w_m \alpha_m = w^T \alpha \)

\[ r = \sqrt{\frac{E}{M}} w^T \alpha s + z \]

\( \Rightarrow \) This is similar to the single-antenna case considered before. For binary case, the decision rule is given as

\[
\text{Re}\left[r(\mathbf{w}^T \alpha)^*\right] \geq 0
\]
Closed-Loop Transmit Diversity (cont’ d)

- Assuming \( s = -\sqrt{E} \)

\[
P\left(\Re\left[z(w^T \alpha)^*\right] \geq \sqrt{\frac{E_s}{M}} |w^T \alpha|^2 |\alpha\right) = ?
\]

\( \Re(z), \Im(z) \sim N(0, N_0/2) \quad \Re\left[z(w^T \alpha)^*\right] \sim N\left(0, \frac{N_0}{2} |w^T \alpha|^2\right) \)

- Using symmetry, the probability of error is found as

\[
P(e|\alpha) = Q\left(\sqrt{\frac{1}{M \frac{E_s}{N_0}}} |w^T \alpha|^2 \right)
\]

- To maximize SNR, the weighting coefficients are chosen as (Ch. 4, p.22 --- Schwarz Inequality)

\[
w = \sqrt{M} \frac{\alpha^*}{\|\alpha\|} \quad \Rightarrow \quad |w^T \alpha|^2 = M \|\alpha\|^2 = M \sum_{m=1}^{M} |\alpha_m|^2
\]

\[
P(e|\alpha) = Q\left(\sqrt{\frac{2E}{N_0} \sum_{m=1}^{M} |\alpha_m|^2} \right) \quad \text{This is the same conditional probability expression obtained for MRC RX diversity!}
\]
Practical Implementation of Closed-Loop Transmit Diversity
Time Division Duplex (TDD)

In TDD, a single radio channel is shared in time so that a portion of the time is used to transmit from BS to the MS and the remaining time is used to transmit from the MS to BS.

Assuming channel reciprocity, the downlink and uplink channels are often assumed to be the same in TDD mode. In this case, the base station can get the necessary feedback information for the computation of weight vector (which will be used for the downlink transmission) from the received signal through the uplink channel.
Practical Implementation of Closed-Loop Transmit Diversity Frequency Division Duplex (FDD)

- In FDD, we have simultaneous radio transmission between the MS and BS. At the BS, separate transmit and receive antennas are used to accommodate the two separate channels. At the MS, a single antenna with duplexer is used. Transmit and receive frequencies are separated by 5% of the nominal frequency for enabling the use of single antenna for simultaneous transmission/reception (Ch.1, p. 8).

- Since uplink and downlink channels are significantly different, mobile station must provide feedback to the base station. For this purpose, in general, a dedicated digital uplink channel is used.
Practical Implementation of Closed-Loop Transmit Diversity

- Closed loop requires feedback information. The possible sources of error in extracting this information are:
  - **The channel estimation error:** Since channel reciprocity does not hold in most practical cases, the performance is sensitive to uplink/downlink differences in TDD. As for FDD, a rapidly time-varying channel will result in unreliable channel estimates.
  - **The quantization error:** Analog channel estimation information will be digitally encoded for transmission.
  - **The feedback delay**
  - **The feedback transmission error** (FDD)

- The difficulties associated with the practical implementation of closed loop transmit diversity hinders its widespread use. The challenge is to design open-loop transmit diversity schemes which will not require weighting at the transmit side, therefore, no feedback.
Open-Loop Transmit Diversity

- Space-Time Coding (STC) was introduced in 1998 and provides a revolutionary solution for the long-standing problem of designing open-loop transmit diversity schemes. It has been already included in 3\(^{rd}\) and 4\(^{th}\) generation standards and for cellular and WLAN.

- STC is able to achieve full spatial diversity (i.e., \(M \times N\)) without channel knowledge at TX, where \(M\) and \(N\) are the numbers of transmit and receive antennas (receive diversity is optional).

- If a symbol goes out of any antenna, it sees \(N\) independent channels, achieving a diversity order of \(N\) (i.e. receive diversity). As discussed before, simply sending the same signal simultaneously from all antennas does not work to extract transmit diversity. For obtaining diversity order of \(M \times N\), symbols must, somehow, be spread across all transmit antennas in a “smart” manner.

- Two general classes of STC: Space-Time Trellis Codes (STTC) and Space-Time Block Codes (STBC). In the following, we will discuss a simple STBC for 2-TX antennas. (Further study of STC is offered in a grad course ECE710 “Lattices and Applications,” Spring 2012)
Space-Time Block Coding for 2-TX Antennas

Alamouti’s Code

➢ Alamouti’s code is the first open-loop transmit diversity technique which provides full diversity with linear processing at the receiver. Note that there have been some earlier examples of transmit diversity schemes which also provide full diversity, but the complexity of receiver structure increases exponentially with the number of TX antennas.

➢ Assume that an $M$-ary modulation scheme is used. Each group $m = \log_2 M$ of information bits is grouped and mapped to modulation symbols. The encoder takes a block of two modulated symbols and maps them to the transmit antennas according to a matrix given by

$$X = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}$$
Alamouti’s Code (cont’d)

- The key feature of the Alamouti’s scheme is that the transmit sequences from each of the antennas are orthogonal. Check that the inner product of the sequences are zero.

\[
\begin{align*}
\text{Time int.:} & \quad 1^{st} \quad 2^{nd} \\
\Downarrow & \quad \downarrow \\
\mathbf{x}_1 &= (x_1, -x_2^*) \\
\Downarrow & \quad \downarrow \\
\mathbf{x}_2 &= (x_2, x_1^*)
\end{align*}
\]

\[
\mathbf{x}_1 \cdot \mathbf{x}_2 = \sum_{i=1}^{2} x_{1,i} x_{2,i}^* = x_1 x_2^* - x_2^* x_1 = 0
\]

- The code matrix has the following property

\[
\mathbf{X}^H \mathbf{X} = \left( |x_1|^2 + |x_2|^2 \right) \mathbf{I}_2
\]

- Orthogonality of the code imposes an “artificial” orthogonality on the channel: in this case, we say that the code orthogonalizes the channel.
Alamouti’s Code (cont’ d)

- Under the assumption that fading coefficients are constant over two symbol periods, the received signal for two consecutive time intervals are given

\[
\begin{align*}
    r_1 &= \alpha_1 x_1 + \alpha_2 x_2 + z_1 \\
    r_2 &= \alpha_1 ( - x_2^*) + \alpha_2 x_1^* + z_2
\end{align*}
\]

- Assume perfect channel state information (CSI) is available at the receiver. The maximum likelihood receiver will choose the pair of signals \( \hat{x}_1, \hat{x}_2 \) which minimize following the distance metric

\[
\begin{align*}
    d^2(r_1, \alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2) + d^2(r_2, - \alpha_1 \hat{x}_2^* + \alpha_2 \hat{x}_1^*) \\
    &= |r_1 - (\alpha_1 \hat{x}_1 + \alpha_2 \hat{x}_2)|^2 + |r_2 - (- \alpha_1 \hat{x}_2^* + \alpha_2 \hat{x}_1^*)|^2
\end{align*}
\]

over all possible values of \((x_1, x_2)\)

- This is in general a \( N \)-dimensional minimization problem. For higher-order modulation schemes and large number of transmit antennas, it might be computationally intensive.
Alamouti’s Code (cont’d)

➢ Let’s consider decoding process again trying to exploit the orthogonal structure of Alamouti’s code

\[
\begin{align*}
    r_1 &= \alpha_1 x_1 + \alpha_2 x_2 + z_1 \\
    r_2^* &= -\alpha_1^* x_2 + \alpha_2^* x_1 + z_2^* \\
\end{align*}
\]

\[
\begin{bmatrix}
    r_1 \\
    r_2^* 
\end{bmatrix} =
\begin{bmatrix}
    \alpha_1 & \alpha_2 \\
    \alpha_2^* & -\alpha_1^* 
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 
\end{bmatrix} +
\begin{bmatrix}
    z_1 \\
    z_2^* 
\end{bmatrix} \rightarrow \mathbf{R} = \mathbf{aX} + \mathbf{Z}
\]

➢ Note the orthogonal structure of channel matrix, i.e.,

\[
\alpha^H \alpha = \left( |\alpha_1|^2 + |\alpha_2|^2 \right) \mathbf{I}_2
\]

➢ Construct the new decision statistics, i.e.,

\[
\tilde{\mathbf{R}} = \alpha^H \mathbf{R} = \alpha^H \mathbf{aX} + \alpha^H \mathbf{Z}
\]

\[
\tilde{\mathbf{R}} = \left( |\alpha_1|^2 + |\alpha_2|^2 \right) \mathbf{I}_2 \mathbf{X} + \tilde{\mathbf{Z}} = \rho \mathbf{I}_2 \mathbf{X} + \tilde{\mathbf{Z}}
\]

➢ A two-dimensional minimization problem is decoupled into two 1-dimensional problems.

\[
\hat{x}_1 = \arg \min_{\hat{x}_1} |\tilde{r}_1 - \rho \hat{x}_1|^2 \quad \hat{x}_2 = \arg \min_{\hat{x}_2} |\tilde{r}_2 - \rho \hat{x}_2|^2
\]
Diversity order = Slope of the performance curve = 2

Alamouti’s scheme achieves the same diversity order as a MRC.

The performance of the Alamouti’s scheme is 3dB worse than MRC. 3dB difference is due to the fact that the energy from each transmit antenna is scaled by 2 to keep the total transmit power fixed.
Receiver Design for Frequency-Selective Channels

- So far, we considered frequency-flat fading channels. As discussed in Chapter 2, frequency-selective channels introduce interference which need to be effectively mitigated at the receiver side.

- Assume that the impulse response of the “equivalent” channel (modeling transmit filter+actual channel+receive filter) is given by $c(t)$. For the transmitted signal $x(t) = \sum_{n=-\infty}^{\infty} x_n \delta(t - nT)$, the received signal is

$$r(t) = x(t) \otimes c(t) + n(t) = \left[ \sum_{n=-\infty}^{\infty} x(t) \delta(t - lT) \right] \otimes c(t) + n(t) = \sum_{l=-\infty}^{\infty} x_l c(t - lT) + n(t)$$

- Sampling the received signal at $t = nT$ and defining $r_n = r(nT), c_n = c(nT)$ and $\eta_n = \eta(nT)$, we write

$$r_n = \sum_{l=-\infty}^{\infty} x_l c_{n-l} + \eta_n = c_0 x_n + \sum_{l=-\infty}^{\infty} x_l c_{n-l} + \eta_n$$

Desired signal term
\[\text{ISI terms}\]
\[\text{Noise term}\]

- ISI severely degrades the performance of transmission accuracy.
Example: Effect of ISI on BER performance of BPSK

- Assume BPSK transmission with the following mapping rule

\[ 0 \Leftrightarrow x = -\sqrt{E}, \quad 1 \Leftrightarrow x = +\sqrt{E} \]

The channel introduces time-dispersion and the received signal is obtained as

\[ r_n = x_n + 0.5x_{n-1} + \eta_n \]

Determine the bit error probability.

- If we assume ISI-free environment, we would obtain \( r_n = x_n + \eta_n \) and BER is simply given as \( P_e = Q\left(\sqrt{2E/N_0}\right) \). However, here we need to consider the effect of previously transmitted symbol. Therefore, we have four possible combinations:

<table>
<thead>
<tr>
<th>( x_{n-1} )</th>
<th>( x_n )</th>
<th>Without ISI</th>
<th>With ISI</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(-\sqrt{E})</td>
<td>(-1.5\sqrt{E})</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>(+\sqrt{E})</td>
<td>(+0.5\sqrt{E})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(-\sqrt{E})</td>
<td>(-0.5\sqrt{E})</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(+\sqrt{E})</td>
<td>(+1.5\sqrt{E})</td>
</tr>
</tbody>
</table>
**Example:** Effect of ISI on BER performance of BPSK (cont’d)

\[ P_b = \sum P(e|x_{n-1}, x_n)P(x_{n-1}, x_n) \]

\[ = \frac{1}{4} \left[ P(e|x_{n-1} = 0, x_n = 0) + P(e|x_{n-1} = 0, x_n = 1) + P(e|x_{n-1} = 0, x_n = 0) + P(e|x_{n-1} = 0, x_n = 1) \right] \]

\[ = \frac{1}{4} \left[ P(-1.5\sqrt{E} + n > 0) + P(0.5\sqrt{E} + n < 0) + P(-0.5\sqrt{E} + n > 0) + P(1.5\sqrt{E} + n < 0) \right] \]

\[ = \frac{1}{2} \left[ Q \left( 1.5 \sqrt{\frac{2E}{N_0}} \right) + Q \left( 0.5 \sqrt{\frac{2E}{N_0}} \right) \right] \]

- At BER=10^{-3}, we need an additional 5.5dB to overcome the ISI.
As the previous example clearly illustrates, we need efficient receiver techniques to mitigate the effects of ISI. One method is to “equalize” the channel.

The optimal equalization technique is maximum likelihood sequence estimation (MLSE). Unfortunately, the complexity of this technique grows exponentially with the length of delay spread.

Equalization techniques can be classified as “linear” and “nonlinear”.

Channel Equalization

- Linear
- Nonlinear

For time-varying channels:
- **Tap Update Algorithms**
  - Transversal
  - LMS
  - RLS
  - Gradient RLS
  - Fast RLS
  - Square-Root RLS

Equalizers

- DFE
- MLSE
Linear Equalization

- We consider a linear equalizer, which is a tapped-delay-line-filter with $2N+1$ taps. The coefficient of the $k^{th}$ tap is denoted by $b_k$.

- The impulse response of the equalizer is $b(t) = \sum_{k=-N}^{N} b_k \delta(t - kT)$

- The transfer function of the equalizer is $B(z) = \sum_{k=-N}^{N} b_k z^{-k}$

- In the following, we will study two linear equalizers: “Zero Forcing (ZF)” and “Minimum Mean Square Error (MMSE)” equalizers which differ from each other by the way equalizer coefficients are determined,
Zero Forcing (ZF) Linear Equalizer

-Ignoring the additive noise term, the received signal is

\[ r(t) = x(t) \otimes c(t) \iff R(z) = X(z)C(z) \]

-The output of the equalizer is \( Y(z) = R(z)B(z) = X(z)C(z)B(z) \)

-Let \( H(z) \) denote the transfer function of the “equalized” system

\[ H(z) = \frac{Y(z)}{X(z)} = C(z)B(z) \]

-Let \( h_n \) denote the impulse response of the “equalized” system

Replacing \( C(z) = \sum_{k=-\infty}^{\infty} c_k z^{-k} \) and \( B(z) = \sum_{k=-N}^{N} b_k z^{-k} \)

\[
\begin{align*}
C(z)B(z) &= \sum_{l=-\infty}^{\infty} \sum_{k=-N}^{N} c_l b_k z^{-(l+k)} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-N}^{N} c_{n-k} b_k z^{-n} \\
&= \sum_{n=-\infty}^{\infty} h_n z^{-n} = H(z) \\
\text{with } n = l + k
\end{align*}
\]

\[
\begin{align*}
h_n &= \sum_{k=-N}^{N} b_k c_{n-k} \\
H(z) &= \sum_{n=-\infty}^{\infty} h_n z^{-n}
\end{align*}
\]
ZF Linear Equalizer

For ISI-free transmission, \( Y(z) = X(z) \). This requires

- \( H(z) = 1 \), i.e. \( B(z) = 1/C(z) \)

or equivalently

- \( h(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \)

This is known as “Nyquist criterion” for ISI-free transmission.
ZF Linear Equalizer (cont'd)

- Given the effective channel $C(z)$, we want to design the equalizer $B(z)$ such that the conditions for “Nyquist criterion” can be satisfied.

- In frequency-domain, we need to design a filter with, $B(z) = 1/C(z)$ which gives us $H(z) = 1$

- In time-domain, we choose the tap coefficients to satisfy the following equation system

\[
\begin{bmatrix}
1 & 0 & ... & 0 \\
0 & 1 & ... & 0 \\
... & ... & ... & ... \\
0 & 0 & ... & 1
\end{bmatrix}
\begin{bmatrix}
\hat{b}_0 \\
\hat{b}_1 \\
\hat{b}_2 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots
\end{bmatrix}
\]

which guarantees $\hat{h}(n) = \begin{cases} 1, & n = 0 \\ 0, & n = \pm 1, \pm 2, \ldots, \pm N \end{cases}$

- For $N \to \infty$, $\hat{h}(n) \to h(n)$
Example: Design of a ZF Equalizer with 3-Taps

- The impulse response of the effective channel is

\[ c(t) = \exp\left(-\frac{|t|}{3T}\right), \quad -\infty < t < \infty \]

where \( T \) is the transmitted symbol interval. Design a 3-tap ZF linear equalizer for the channel.

- The discrete-time representation of the channel

\[ c_n = c(t)_{|t=nT} = \exp\left(-\frac{|nT|}{3T}\right) = \exp\left(-\frac{|n|}{3}\right) \]

- The “equalized system” impulse response

\[ h_n = b_n \otimes c_n = \sum_{k=-1}^{1} b_k c_{n-k} \]

We want to determine \( b_{-1}, b_0, b_1 \) such that

\[ h(n) = \begin{cases} 1, & n = 0 \\ 0, & n = \pm 1 \end{cases} \]
Example: Design of a ZF Equalizer with 3-Taps (cont’d)

We need to solve the following system of equations:

\[
\begin{bmatrix}
    c_0 & c_{-1} & c_{-2} \\
    c_1 & c_0 & c_{-1} \\
    c_2 & c_1 & c_0
\end{bmatrix}
\begin{bmatrix}
    b_{-1} \\
    b_0 \\
    b_1
\end{bmatrix}
= 
\begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix}
\]

This yields

\[
\begin{bmatrix}
    b_{-1} \\
    b_0 \\
    b_1
\end{bmatrix}
= 
\begin{bmatrix}
    1 & \exp(-1/3) & \exp(-2/3) \\
    \exp(-1/3) & 1 & \exp(1/3) \\
    \exp(-2/3) & \exp(-1/3) & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix}
= 
\begin{bmatrix}
    -1.4726 \\
    3.1103 \\
    -1.4726
\end{bmatrix}
\]
Example: Effect of Noise on the Performance of ZF Equalizer

- Assume BPSK transmission with the following mapping rule
  \[ 0 \iff x = -\sqrt{E}, \ 1 \iff x = +\sqrt{E} \]

The channel introduces time-dispersion and the received signal is obtained as

\[ r_n = x_n + 0.5x_{n-1} + \eta_n \]

Assuming an ideal (i.e. infinite taps) ZF equalizer is employed at the receiver side, determine the bit error probability.

- The impulse response and transfer function of the effective channel are given by
  \[ c(t) = \delta(t) + 0.5\delta(t-T) \iff C(z) = \sum_{k=-\infty}^{\infty} c_k z^{-k} = 1 + 0.5z^{-1} \]

- To ideally equalize the channel, the transfer function of the equalizer should be
  \[ B(z) = \frac{1}{C(z)} = \frac{1}{1 + 0.5z^{-1}} = \sum_{n=0}^{\infty} (-0.5z^{-1})^n \text{ for } |0.5z^{-1}| < 1 \]
Ex: Effect of Noise on the Performance of ZF Equalizer (cont’d)

- By inverse z-transform, we have \( b_n = (-0.5)^n u_n \) \( u \): unit step function

- The equalized output \( y_n = r_n \otimes b_n = [x_n \otimes c_n + \eta_n] \otimes b_n \)
  \[ = x_n \otimes c_n \otimes b_n + \eta_n \otimes b_n \]

- The noise term at the equalizer output

\[ v_n = b_n \otimes \eta_n = \sum_{k=0}^{\infty} (-0.5)^k \eta_{n-k} \]

- \( \eta_n \) is zero-mean Gaussian with \( \sigma_{\eta}^2 = N_0/2 \). \( v_n \) is also Gaussian

\[ E[v_n] = \sum_{k=0}^{\infty} (-0.5)^k E[\eta_{n-k}] = 0 \]

\[ E[v_n^2] = \sum_{k=l} (\sigma_{\eta}^2)^2 E[\eta_{n-k}^2] + \sum_{k \neq l} \sum_{k=0}^{\infty} (-0.5)^k (-0.5)^l E[\eta_{n-k} \eta_{n-l}] = \frac{4 N_0}{3} \frac{2 N_0}{2} = \frac{2 N_0}{3} \]

- The bit error probability is

\[ Q\left(\sqrt{\frac{E}{\sigma_{\eta}^2}}\right)_{\sigma_{\eta}^2 = 2 N_0/3} = Q\left(\sqrt{\frac{3 E}{2 N_0}}\right) \]
Equalization improves the performance. However, it increases the noise variance as well. Therefore, the equalized system does not have a transmission accuracy as good as the system without ISI.
Minimum Mean Square Error (MMSE) Linear Equalizer

- In ZF equalizer design, the effect of noise component is ignored. In equalizer design, we should consider the performance degradation due to both ISI and channel noise.

- One method is to choose the equalizer coefficients such that MSE is minimized. The equalizer maximizes the signal-to-distortion ratio at the equalizer output.

- The output of the equalizer \( y_n = b_n \otimes r_n = \sum_{k=-N}^{N} b_k r_{n-k} \)

- The equalization error is defined as the difference between the desired output \( x_n \) and the actual output \( y_n \), i.e. \( \varepsilon_n = x_n - y_n \)

- The problem of choosing the equalizer tap coefficients is to solve the following minimization problem:

\[
\min_{\mathbf{b}} E[\varepsilon_n^2]
\]

where \( \mathbf{b} = (b_{-N}, \ldots, b_{-1}, b_0, b_1, \ldots, b_N)^T \) represents the linear equalizer of \( 2N+1 \) taps.
The MMSE solution is obtained by first differentiating $e$ with respect to the equalizer coefficients $b_k$, $k = 0, \pm 1, \pm 2, \ldots$ and then setting the derivatives to zero.

$$
\frac{\partial e}{\partial b_k} = 0 \Rightarrow \sum_{l=-N}^{N} b_l R_r(k-l) = R_{xr}(k) \quad k = 0, \pm 1, \pm 2, \ldots
$$

where we define $R_r(k-l) = E[r_{n-l}r_{n-k}]$ and $R_{xr}(k) = E[x_n r_{n-k}]$.

In matrix form, $\mathbf{R}_r \mathbf{b} = \mathbf{R}_{xr} \Rightarrow \mathbf{b} = \mathbf{R}_r^{-1} \mathbf{R}_{xr}$, which is known as the Wiener-Hopf equation.
MMSE Linear Equalizer (cont’ d)

➢ In practice, the autocorrelation matrix and the correlation matrix are unknown a priori. For a time-invariant channel, a test signal can be transmitted over the channel and a time-average estimate can be obtained, i.e.

\[
\hat{R}_r(k-l) = \frac{1}{\tilde{N}} \sum_{n=1}^{\tilde{N}} r_{n-l}r_{n-k} \quad \hat{R}_{xr}(k) = \frac{1}{\tilde{N}} \sum_{n=1}^{\tilde{N}} x_n r_{n-k}
\]

where \( \tilde{N} \) is the number of data samples in the test signal.

➢ For a time-varying channel, the equalizer coefficients should change according to the channel status to track the channel variations \( \rightarrow \) Adaptive equalizer.

➢ We can use the following iterative computation to update the tap coefficients

\[
b_k(n+1) = b_k(n) + \alpha \cdot \Delta b_k(n) \quad k = 0, \pm 1, \pm 2, \ldots, \pm N
\]

\( \alpha \) : Step size
\( \Delta b_k \): Direction determining parameter
MMSE Linear Equalizer (cont’d)

- The gradient \( \partial e/\partial b_k \) can be used as a direction parameter.

\[
e = E\left[\varepsilon_n^2\right] \quad \text{where}
\]

\[
\varepsilon_n = x_n - y_n = x_n - \sum_{k=-N}^{N} b_k r_{n-k}
\]

\[
\frac{\partial e}{\partial b_k} = E\left[2\varepsilon_n \frac{\partial \varepsilon_n}{\partial b_k}\right] = -2E[\varepsilon_n r_{n-k}]
\]

- For simplicity, we can use the instantaneous value, i.e. \( \varepsilon_n r_{n-k} \)

\[
b_k(n+1) = b_k(n) + \alpha(\varepsilon_n r_{n-k})
\]

- This iterative algorithm is called the least-mean-square (LMS) algorithm for tap coefficients.
Decision Feedback Equalizer (DFE)

- Noise enhancement is the major drawback of linear equalizer.

- The optimal equalizer is MLSE (non-linear technique), which can be implemented in practice by Viterbi-algorithm. However, its complexity (given in terms of number of states) grows exponentially with channel memory.

- Decision Feedback Equalizer is a non-linear technique with low complexity.

Basic idea of DFE:

- Recall that the received signal subject ISI is given as

\[
r_n = \sum_{l=-\infty}^{\infty} c_l x_n - l = c_0 x_n + \sum_{l=-\infty}^{1} c_l x_{n-l} + \sum_{l=1}^{\infty} c_l x_{n-l} + \eta_n
\]

(1) Function of the “precursors” of the channel impulse response \( \{c_{-\infty},...,c_{-2},c_{-1}\} \)

(2) Function of the “postcursors” of the channel impulse response \( \{c_{1},c_{2},...,c_{\infty}\} \) which is also related to the previously transmitted symbols.

- Assuming no detection errors, the information of the previous symbols is available to the receiver, which can use it to combat ISI.
**DFE (cont’d)**

- We consider a DFE which has a \((N+1)\)-tap feedforward filter and an \(M\)-tap feedback filter.
- The input to the feedforward filter is the received signals.
- The input to the feedback filter is the detected previous symbols.
- If past decisions are assumed to be correct, the ISI contributed by these symbols can be canceled exactly, by subtracting the past symbol values with appropriate weighting.
DFE (cont’d)

- Here, we study a particular version of the DFE known as zero forcing DFE.
- Transfer function of the channel  \( C(z) = \sum_{k=-\infty}^{\infty} c_k z^{-k} \)
- Transfer function of the feedforward filter  \( B(z) = \sum_{k=-N}^{0} b_k z^{-k} \)
- Let \( H(z) \) denote the transfer function of the “equalized” system
  \[
  H(z) = C(z)B(z) = \sum_{l=-\infty}^{\infty} \sum_{k=-N}^{0} c_l b_k z^{-(l+k)} = \sum_{n=-\infty}^{\infty} \sum_{k=-N}^{0} c_{n-k} b_k z^{-n} = \sum_{n=-\infty}^{\infty} h_n z^{-n}
  \]

  where  \( h_n = \sum_{k=-N}^{0} b_k c_{n-k} \)

- The tap coefficients \( b_k \) of the feedforward filter should be chosen in such a way that \( h_n = 0 \) for \( n < 0 \) (i.e. Nyquist criterion) without changing the relative values of remaining components for \( n > 0 \).

\[
\begin{bmatrix}
  c_0 & c_{-1} & \ldots & c_{-N} \\
  c_1 & c_0 & \ldots & c_{-N+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_N & c_{N-1} & \ldots & c_0
\end{bmatrix}
\begin{bmatrix}
  b_{-N} \\
  b_{-N+1} \\
  \vdots \\
  b_0
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  1
\end{bmatrix}
\]
DFE (cont’d)

- The tap coefficients of the feedback filter \( d_j \) are chosen to be \( h_j, j=1,2\ldots M \) so that the effect of postcursors can be removed by the feedback filter.

- The input to the decision device

\[
y_n = \sum_{k=-N}^{0} b_k r_{n-k} - \sum_{j=1}^{M} d_j \hat{x}_{n-j} \quad d_j = h_j
\]

- Under the assumption that ISI due to the precursors is completely removed, the output of the feedforward filter is approximated as

\[
\sum_{k=-N}^{0} b_k r_{n-k} \approx x_n + \sum_{j=1}^{M} h_j x_{n-j} + \nu_n \quad \text{where} \quad \nu_n = \sum_{k=-N}^{0} b_k \eta_{n-k}
\]

- If we assume that the previous symbols are correctly detected,

\[
\hat{x}_n = D(y_n) = D \left[ \left( x_n + \sum_{j=1}^{M} h_j x_{n-j} + \nu_n \right) - \left( \sum_{j=1}^{M} h_j \hat{x}_{n-j} \right) \right] = D(x_n + \nu_n)
\]

where \( D \) denotes the decision function.
DFE (cont’d)

- The received noise component $\eta_n$ is zero-mean Gaussian with $\sigma^2_n = N_0/2$.
- The noise term in the decision device is $v_n = \sum_{k=-N}^{0} b_k \eta_{n-k}$

\[
E[v_n] = \sum_{k=-N}^{0} b_k E[\eta_{n-k}] = 0
\]

\[
E[v_n^2] = \sum_{k=-N}^{0} b_k^2 E[\eta_{n-k}^2] + \sum_{k \neq l} b_k^2 b_{k+l} E[\eta_{n-k}\eta_{n-l}] = 0 = \frac{N_0}{2} \sum_{k=-N}^{0} b_k^2
\]

- Compared with the ZF linear equalizer, ZF-DFE in general reduces the noise amplification effect.
- DFE suffers from the inherent “error propagation” problem due to the fact that the feedback filter uses the detected symbols. When a particular incorrect decision is fed back, the DFE output reflects this error during the next few symbols as the incorrect decision traverses the feedback delay line.
Multicarrier Modulation

- One way to combat the ISI caused by a frequency-selective channel is to divide the wideband channel into several smaller sub-channels and send data in parallel down the sub-channels.

- If enough parallel sub-channels are used, each of them behaves practically as a frequency-flat channel, eliminating the need for an equalizer.

- Multicarrier communication was first used for military HF radios in late 1950’s. Starting around 1990, it has been used in many diverse wired and wireless applications including
  - Digital audio and video broadcasting (DAB/DVB) in Europe
  - Digital subscriber line (DSL)
  - IEEE802.11a/g Wireless LANs
  - Also a candidate technology for next generation cellular systems
Multicarrier Modulation (cont’ d)

- Consider a system with data rate of $R$ and bandwidth of $B$. Let $N$ be the number of sub-channels. Therefore, each sub-channel has a bandwidth of $B_N = B/N$.

- Assume that $B_c$ is the coherence bandwidth of the channel. By proper choice of $N$, we can ensure that $B_c << B_N$ leading to a frequency-flat assumption for each sub-channel.

- The transmitted signal has the form of

$$s(t) = \sum_{i=0}^{N-1} s_i g(t) \cos(2\pi f_i t + \phi_i)$$

where

- $f_i = f_0 + iB_N$, $i = 0,1,2...N - 1$
- $s_i$: Modulation symbol associated with the $i^{th}$ subcarrier
- $\phi_i$: Phase offset of the $i^{th}$ subcarrier
- $g(t)$: Pulse shape function
At the receiver side, we employ lowpass filters which are centered around subchannel carrier frequency so that each signal can be separated.
Example: A Multicarrier System with Raised-Cosine Pulse

Consider a multicarrier system with $T_N=0.2\text{ms}$. Assume the system has $N=128$ subchannels. If raised cosine pulses with $\beta=1$ are used and the additional bandwidth due to time-limiting required to ensure minimal power outside the signal bandwidth is $\varepsilon/T_N$, what is the total bandwidth of the system?

\[
B = \frac{N(1 + \beta + \varepsilon)}{T_N} = \frac{128(1+1+0.1)}{0.2 \cdot 10^{-3}} = 1.344\text{MHz}
\]
Multicarrier Modulation (cont’ d)

- The previous implementation requires near-ideal (therefore expensive) lowpass filters and \( N \) independent modulators/demodulators which entails significant expense, size and power consumption.

- Now, we will consider an alternative and more efficient implementation which allows subcarriers to overlap and removes the need for tight filtering.

- Consider the subcarriers \( \cos(2\pi (f_0 + i/T_N )t + \phi_i) \) which are separated from each other by \( 1/T_N \). They form a set of orthogonal basis functions since

\[
\frac{T_N}{0} \int \cos(2\pi (f_0 + i/T_N )t + \phi_i) \cos(2\pi (f_0 + j/T_N )t + \phi_j) dt
\]

\[
= \frac{1}{2} \int_{0}^{T_N} \cos(2\pi (i - j)t/T_N + \phi_i - \phi_j) dt + \frac{1}{2} \int_{0}^{T_N} \cos(4\pi f_0 + 2\pi (i + j)t/T_N + \phi_i + \phi_j) dt
\]

\[
= \frac{1}{2} T_N \delta(i - j)
\]
Multicarrier Modulation (cont’d)

Since the subchannels overlap, the receiver needs to be modified.

\[
\hat{s}_i = \frac{2}{T_N} \int_0^{T_N} \left( \sum_{j=0}^{N-1} s_j \cos(2\pi f_j t + \phi_j) \right) g(t) \cos(2\pi f_i t + \phi_i) dt
\]

\[
= \frac{2}{T_N} \sum_{j=0}^{N-1} s_j \int_0^{T_N} \cos(2\pi (f_0 + j/T_N) t + \phi_j) \cos(2\pi (f_0 + i/T_N) t + \phi_i) dt
\]

\[
= \sum_{j=0}^{N-1} s_j \delta(j - i) = s_i
\]
Multicarrier Modulation (cont’ d)

- As the subchannels are allowed to overlap, the required bandwidth will be reduced.
- Assume that we use raised cosine pulse shape with roll-off factor $\beta$. The total bandwidth will be given as

$$B = \frac{1}{T_N} (N - 1) + \frac{1 + \beta + \varepsilon}{T_N} = \frac{N + \beta + \varepsilon}{T_N}$$

- For the same parameters in the previous example, the required bandwidth is

$$B = \frac{N + \beta + \varepsilon}{T_N} = \frac{128 + 1 + 0.1}{0.2 \cdot 10^{-3}} = 645.5\text{kHz} \text{ (nearly half of the previous bandwidth)}$$
Orthogonal Frequency Division Multiplexing (OFDM)

- OFDM provides a discrete implementation of multicarrier modulation which eliminates the need for multiple modulators/demodulators.

- Assume the transmission of \( \sum_{k=0}^{\infty} a_{k,n} \delta(t - kT) \) where

  \( a_{k,n} \): Modulation symbol to be sent over the \( k^{th} \) symbol interval \( t \in [kT, (k + 1)T] \) in the \( n^{th} \) subband \( n = 0,1,2...N - 1 \)

- The complex envelope of the transmitted OFDM signal has the form of

  \[
  v(t) = \sum_{k=0}^{\infty} \sum_{n=0}^{N-1} a_{k,n} \varphi(t - kT)
  \]

  where \( \{\varphi_n(t)\}, n = 0,1,2...N - 1 \) is a set of complex orthonormal waveforms

  \[
  \varphi_n(t) = \begin{cases} 
  \exp \left[ j 2\pi \left( \frac{2n - (N - 1)}{2T} \right) t \right], & t \in [0,T] \\
  0, & t \notin [0,T]
  \end{cases}
  \]
For simplicity, consider $k=0$

$$v(t) = \sum_{n=0}^{N-1} a_{0,n} \exp\left( j \frac{2\pi nt}{NT_N} \right) \exp\left[ -j \frac{\pi (N-1)t}{NT_N} \right] \quad 0 \leq t \leq NT_N$$

Noting that the term $\exp\left[ -j \pi (N-1)t/(NT_N) \right]$ is not a function of $n$ and can be combined with the carrier term $\exp\left[j2\pi f_c t \right]$. The complex envelope can thus be written as

$$v(t) = \sum_{n=0}^{N-1} a_{0,n} \exp\left( j \frac{2\pi nt}{NT_N} \right) \quad 0 \leq t \leq NT_N$$

with the corresponding carrier $\exp\left[j2\pi (f_c - (N-1)/(2NT_s))t \right]$

Sampling $v(t)$ at $t = lT_N$ yields

$$A_{0,l} \overset{\text{def}}{=} v(lT_N) = \sum_{n=0}^{N-1} a_{0,n} \exp\left( j \frac{2\pi nl}{N} \right) \quad l = 0,1,...,N-1$$

which is the inverse discrete Fourier transform (IDFT) of $\{a_{0,N}\}$. 
OFDM (cont’ d)

- By proper choice of waveforms, the transmitter and receiver of the multicarrier system can be implemented through IDFT/DFT pair.

- The input information sequence is partitioned into blocks of length $N$, each of which is given by
  \[
  (a_{k,0}, a_{k,1}, ..., a_{k,N-1}) \quad k = 0,1,2,...
  \]

- After D/A conversion and carrier modulation, the transmitted OFDM signal is
  \[
  x(t) = \text{Re}\left\{v(t)\exp\left[j2\pi \left(f_c - \frac{N-1}{2NT_N}\right)t\right]\right\}
  \]

- At the receiver side (after down-conversion to baseband and A/D conversion), the received samples are \((r_{0,0}, r_{0,1}, ..., r_{0,N-1})\) for $k=0$. Demodulation is performed by performing DFT, which produces $N$ decision variables $z_{0,n}$, $n = 0,1,...,N - 1$
  \[
  z_{0,n} = \frac{1}{N} \sum_{l=0}^{N-1} r_{0,l} \exp\left(-j \frac{2\pi nl}{N}\right)
  \]