ECE 602 – Introduction to Optimization

Home Assignment 1

Due: February 12, 2024

Exercise 1 (Gradient)

Let $x \in \mathbf{R}^n$ and $A \in \mathbf{R}^{m \times n}$. Also, let $f : \mathbf{R}^n \to \mathbf{R}$ be defined according to

$$f(x) = \sum_{i=1}^{m} \sqrt{(Ax)_i^2 + \epsilon},$$

where $(Ax)_i$ denotes the ith element of Ax and $0 < \epsilon \ll 1$ is a small number. Find the gradient of $f(x)$ using its *external definition*.

Exercise 2 (Convexity)

Explain which of the following sets are convex. Show your work.

a) The sublevel set of a convex function f, i.e., $C_\alpha = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$.

b) The set of positive semidefinite matrices $\mathbf{S}_+^n$.

Explain which of the following functions are convex. Show your work.

a) $f(x) = \frac{1}{2} x^T Q x + c^T x$, where $Q \in \mathbf{S}_+^n$ and $c \in \mathbf{R}^n$.

b) $f(x) = g(h(x))$ where $h : \mathbb{R}^n \to \mathbb{R}$ is convex, while $g : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing

Exercise 3 (Global minimum of convex functions)

Assume that U is a convex subset of a normed linear space E. Prove that the set of all *global* minimizers of f is convex. [*Hint:* Use the result in Exercice (2.a).]

Exercise 4 (Dual norms)

Prove the following statements:

a) The dual norm of $||x||_1$ is $||x||_\infty$.

b) The dual norm of $||x||_2$ is $||x||_2$.

Exercise 5

Consider the following optimization problem in $\mathbf{R}^2$:

$$\begin{array}{ll} \text{minimize} & f(x) = (x_2 - x_1)^2 + (1 - x_1)^2 \\ \text{subject to} & ||x||_1 \leq 1. \end{array}$$

We will find a solution to this problem using the following steps:

a) Find the gradient ∇f and the Hessian $\nabla^2 f$ of f.

b) Discuss the convexity of f and of the constraint function.

c) Find the minimizer of the unconstrained problem $\min f(x)$. Does the solution satisfy the constraint?

d) If the above solution is infeasible, find a solution to the constrained optimization problem on the boundary of the feasible region. [*Hint:* Validate your solution based on the result of Exercise (3.a).]

e) Finally, using either MATLAB or PYTHON, draw a contour plot of $f(x)$ along with the feasible region to verify your solution.