Home Assignment №4

Due on April 8, 2024

Exercise 1

What is the solution of the norm approximation problem with one scalar variable $x \in \mathbf{R}$

$$\text{minimize } \|\mathbf{1}x - b\|$$

for the ℓ_1-, ℓ_2-, and ℓ_∞-norms? Explain your answers.

Exercise 2

An *interval matrix* in $\mathbf{R}^{m\times n}$ is a matrix whose entries are intervals:

$$\mathcal{A} = \left\{ A \in \mathbf{R}^{m\times n} \mid |A_{ij} - \bar{A}_{ij}| \le R_{ij},\ i = 1,\dots,m,\ j = 1,\dots,n \right\}$$

The matrix $\bar{A} \in \mathbf{R}^{m\times n}$ is called the *nominal value* or *centre value*, and $R \in \mathbf{R}^{m\times n}$, which is element-wise nonnegative, is called the *radius*.

The *robust least-squares problem*, with interval matrix, is

$$\min_x \sup_{A\in\mathcal{A}} \|Ax - b\|_2,$$

The problem data are $\mathcal{A}$ (i.e., $\bar{A}$ and R) and $b \in \mathbf{R}^m$. The objective, as a function of x, is called *the worst-case residual norm*. The robust least-squares problem is evidently a convex optimization problem.

Formulate the interval matrix robust least-squares problem as a standard optimization problem, e.g., a QP, SOCP, or SDP. You can introduce new variables if needed. Your reformulation should have a number of variables and constraints that grows linearly with m and n, and not exponentially.

Exercise 3

Using MATLAB, generate the following test signal of length 256:

```
m=256;
t=linspace(0,1,m)';
y=exp(-128*((t-0.3).^2))-3*(abs(t-0.7).^0.4);
```

Also, generate the following matrix of size 256×512:

```
mpdict=wmpdictionary(m,'LstCpt',{'dct',{'wpsym4',2}});
A=full(mpdict);
```

The columns of A consist of two orthogonal bases commonly used in signal analysis, viz. a discrete cosine transform basis (the first 256 columns) and a wavelet basis (the last 256 columns). Our goal is to represent y in terms of the columns of A, i.e., to find x such that $Ax = y$. Needless to say, since the columns of A are linearly dependent, there is no unique way to achieve the above goal. To overcome this difficulty, we consider the following norm minimization problem

$$\min_x \ \|x\| \quad \text{subject to } Ax = y$$

which finds the "smallest" x among all possible solutions of $Ax = b$.

(a) Find solutions to the above problem for the case of ℓ_2- and ℓ_1-norm. In particular, in the case of ℓ_1-norm, cast the problem as an LP and then solve it using CVX.

(b) Using the above solutions (which we denote by x_2 and x_1, respectively), reconstruct their corresponding approximations of y. How close are they to the original signal y? (You may want to use $\|A{*}x_i{-}y\|_2^2/\|y\|_2^2$, $i = 1, 2$, as a measure of relative error.)

(c) Modify x_1 and x_2 by keeping only 5% of their largest (in absolute value) entries, while setting the rest of their entries to zero. What is the accuracy of your reconstructions now in both cases? Plot these reconstructions overlapped over the original y and indicate the relative errors.

(d) Repeat the previous experiment, while keeping 3% and then 1% of the largest (in absolute value) entries of x_1 and x_2. What percentage of the entries of x_2 should you keep to (approximately) reach the same relative error as in the case of x_1 with 3% "compression rate".