University of Waterloo
Department of Electrical and Computer Engineering
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FINAL EXAMINATION
ECE 602

Surname								
Legal Given Name(s)								
UW Student ID Number								

Instruction:

1. There are 100 marks.
2. This is a written, closed-book exam. Please turn off all electronic media and store them under your desk.
3. Be neat. Poor presentation will be penalized.
4. No questions will be answered during the exam. If in doubt, state your assumption(s) and continue.
5. Do not leave during the examination period.
6. Do not stand up until all exams have been picked up.

Question 1 (10 points)

Let $C \subset \mathbf{R}^n$ be the solution set of a quadratic inequality

$$C = \left\{x \in \mathbf{R} \mid x^T A x + b^T x + c \leq 0\right\}$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}$, and $c \in \mathbf{R}$. Show that C is convex if $A \succeq 0$. Is the converse of this statement true?
Hint: A set is convex if and only if its intersection with an arbitrary line is convex.

Question 2 (15 points)

For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

1. $f(x) = e^x - 1$ on $\mathbf{R}$.
2. $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.
3. $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.
4. $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.
5. $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

Question 3 (10 points)

Prove that if f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then $f \cdot g$ is convex.

Question 4 (10 points)

1. Define $g(x) = f(x) + c^T x + d$, where f is convex. Express g^* in terms of f^* (and c, d).
2. Express the conjugate of the perspective of a convex function f in terms of f^*.

Question 5 (10 points)

Consider an LP in inequality form,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, 2, \ldots, m, \end{array}$$

in which there is some uncertainty in the parameters a_i. In particular, a_i are known to lie in given ellipsoids $a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$, where $P_i \in \mathbf{R}^{n \times n}$. Express this robust LP as a SOCP.

Question 6 (10 points)

Prove that $x^* = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & -1 \leq x_i \leq 1, \quad i = 1, 2, 3, \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}, \quad r = 1.$$

Question 7 (15 points)

Formulate the following problems as LPs.

1. $\min \|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$.
2. $\min \|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$.
3. $\min \|Ax - b\|_1 + \|x\|_\infty$.

In each problem, $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ are given.

Question 8 (10 points)

Derive a Lagrange dual for the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} \phi(r_i) \\ \text{subject to} & r = Ax - b, \end{array}$$

where

$$\phi(u) = \begin{cases} u^2, & |u| \leq 1 \\ 2|u| - 1, & |u| > 1. \end{cases}$$

Question 9 (10 points)

Consider the problem

$$\text{minimize } f(x) = \sum_{i=1}^{n} \psi(x_i - y_i) + \lambda \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2,$$

where $\lambda > 0$ is a smoothing parameter, ψ is a convex penalty function, and $x \in \mathbf{R}^n$ is the variable. What is the structure of the Hessian of f?

QUESTION 1

Solution. A set is convex if and only if its intersection with an arbitrary line $\{\hat{x}+tv \mid t \in \mathbf{R}\}$ is convex.

(a) We have

$$(\hat{x}+tv)^T A(\hat{x}+tv) + b^T(\hat{x}+tv) + c = \alpha t^2 + \beta t + \gamma$$

where

$$\alpha = v^T A v, \qquad \beta = b^T v + 2\hat{x}^T A v, \qquad \gamma = c + b^T \hat{x} + \hat{x}^T A \hat{x}.$$

The intersection of C with the line defined by $\hat{x}$ and v is the set

$$\{\hat{x}+tv \mid \alpha t^2 + \beta t + \gamma \leq 0\},$$

which is convex if $\alpha \geq 0$. This is true for any v, if $v^T A v \geq 0$ for all v, *i.e.*, $A \succeq 0$. The converse does not hold; for example, take $A = -1$, $b = 0$, $c = -1$. Then $A \not\succeq 0$, but $C = \mathbf{R}$ is convex.

QUESTION 2

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.
Solution. Strictly convex, and therefore quasiconvex. Also quasiconcave but not concave.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.
Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is neither positive semidefinite nor negative semidefinite. Therefore, f is neither convex nor concave. It is quasiconcave, since its superlevel sets

$$\{(x_1, x_2) \in \mathbf{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$$

are convex. It is not quasiconvex.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.
Solution. The Hessian of f is

$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2/(x_1^2) & 1/(x_1 x_2) \\ 1/(x_1 x_2) & 2/x_2^2 \end{bmatrix} \succeq 0$$

Therefore, f is convex and quasiconvex. It is not quasiconcave or concave.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.
Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$

which is not positive or negative semidefinite. Therefore, f is not convex or concave. It is quasiconvex and quasiconcave (*i.e.*, quasilinear), since the sublevel and superlevel sets are halfspaces.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.
Solution. f is convex, as mentioned on page 72. (See also figure 3.3). This is easily verified by working out the Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} 2/x_2 & -2x_1/x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix} = (2/x_2) \begin{bmatrix} 1 \\ -2x_1/x_2 \end{bmatrix} \begin{bmatrix} 1 & -2x_1/x_2 \end{bmatrix} \succeq 0.$$

Therefore, f is convex and quasiconvex. It is not concave or quasiconcave (see the figure).

QUESTION 3

Solution.

(a) We prove the result by verifying Jensen's inequality. f and g are positive and convex, hence for $0 \leq \theta \leq 1$,

$$\begin{aligned} f(\theta x + (1-\theta)y)\, g(\theta x + (1-\theta)y) &\leq (\theta f(x) + (1-\theta)f(y))\,(\theta g(x) + (1-\theta)g(y)) \\ &= \theta f(x)g(x) + (1-\theta)f(y)g(y) \\ &\quad + \theta(1-\theta)(f(y) - f(x))(g(x) - g(y)). \end{aligned}$$

The third term is less than or equal to zero if f and g are both increasing or both decreasing. Therefore

$$f(\theta x + (1-\theta)y)\, g(\theta x + (1-\theta)y) \leq \theta f(x)g(x) + (1-\theta)f(y)g(y).$$

QUESTION 4

(a) *Conjugate of convex plus affine function.* Define $g(x) = f(x) + c^T x + d$, where f is convex. Express g^* in terms of f^* (and c, d).

Solution.

$$\begin{aligned} g^*(y) &= \sup(y^T x - f(x) - c^T x - d) \\ &= \sup((y-c)^T x - f(x)) - d \\ &= f^*(y-c) - d. \end{aligned}$$

(b) *Conjugate of perspective.* Express the conjugate of the perspective of a convex function f in terms of f^*.

Solution.

$$\begin{aligned} g^*(y,s) &= \sup_{x/t \in \mathbf{dom}\, f, t>0} (y^T x + st - tf(x/t)) \\ &= \sup_{t>0} \sup_{x/t \in \mathbf{dom}\, f} (t(y^T(x/t) + s - f(x/t))) \\ &= \sup_{t>0} t(s + \sup_{x/t \in \mathbf{dom}\, f} (y^T(x/t) - f(x/t))) \\ &= \sup_{t>0} t(s + f^*(y)) \\ &= \begin{cases} 0 & s + f^*(y) \le 0 \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

QUESTION 5

See the textbook.

QUESTION 6

Solution. We verify that x^* satisfies the optimality condition (4.21). The gradient of the objective function at x^* is

$$\nabla f_0(x^*) = (-1, 0, 2).$$

Therefore the optimality condition is that

$$\nabla f_0(x^*)^T (y - x) = -1(y_1 - 1) + 2(y_2 + 1) \ge 0$$

for all y satisfying $-1 \le y_i \le 1$, which is clearly true.

QUESTION 7

(c) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -\mathbf{1} \leq x \leq \mathbf{1}, \end{array}$$

with variables $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

(d) Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \leq x \leq y \\ & -\mathbf{1} \leq Ax - b \leq \mathbf{1} \end{array}$$

with variables x and y.

(e) Equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y + t \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -t\mathbf{1} \preceq x \preceq t\mathbf{1}, \end{array}$$

with variables x, y, and t.

QUESTION 8

Solution. We first derive a dual for general penalty function approximation. The Lagrangian is

$$L(x, r, \lambda) = \sum_{i=1}^{m} \phi(r_i) + \nu^T (Ax - b - r).$$

The minimum over x is bounded if and only if $A^T \nu = 0$, so we have

$$g(\nu) = \begin{cases} -b^T \nu + \sum_{i=1}^m \inf_{r_i} (\phi(r_i) - \nu_i r_i) & A^T \nu = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Using

$$\inf_{r_i} (\phi(r_i) - \nu_i r_i) = - \sup_{r_i} (\nu_i r_i - \phi(r_i)) = -\phi^*(\nu_i),$$

we can express the general dual as

$$\begin{array}{ll} \text{maximize} & -b^T \nu - \sum_{i=1}^m \phi^*(\nu_i) \\ \text{subject to} & A^T \nu = 0. \end{array}$$

Huber penalty.

$$\phi^*(z) = \begin{cases} z^2/4 & |z| \le 2 \\ \infty & \text{otherwise,} \end{cases}$$

so we get the dual problem

$$\begin{array}{ll} \text{maximize} & -(1/4)\|\nu\|_2^2 - b^T\nu \\ \text{subject to} & A^T\nu = 0 \\ & \|\nu\|_\infty \le 2. \end{array}$$

QUESTION 9

Tridiagonal.