University of Waterloo
Department of Electrical and Computer Engineering
Winter, 2017

ECE 602: Introduction to Optimization

FINAL EXAMINATION

Surname								
Legal Given Name(s)								
UW Student ID Number								

Instruction:

1. There are 100 points in total.
2. This is a written, open-book exam (only unannotated lecture slides are allowed). Please turn off all electronic media and store them under your desk.
3. Be neat. Poor presentation will be penalized.
4. **No questions will be answered during the exam**. If in doubt, state your assumption(s) and continue.
5. Do not leave during the examination period without permission.
6. Do not stand up until all the exams have been picked up.

Do well!

Question 1 (15 points)

Consider the non-linear optimization problem

$$\begin{aligned} \min \quad & \exp(2x_1) + \exp(-2x_2) \\ \text{s.t.} \quad & x_1^2 + x_2^2 - 3x_1 - 4x_2 \leq 0, \\ & x_1 + x_2 = 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

a) Is the problem a convex optimization problem or not?

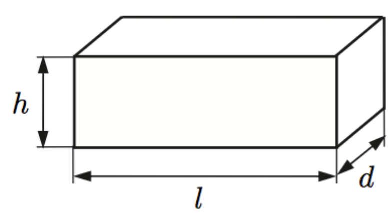
b) Apply Slater's constraint qualification to show if the problem admits strong duality.

c) Give the Lagrange function and the Lagrange dual for this problem.

d) Derive the KKT conditions for the given problem.

Question 2 (15 points)

You have to design a 3-D block as a water storage. The volume of the block has to be at least 9 cubic meters. The base area of the block is at most 6 square meters, while the height of the block is at least 1 and at most 2 meters. Your task is to design the block that satisfies the above conditions and the difference between the base area and the height of the block is maximal.



a) Model the above problem as a constrained (non-linear) optimization problem.

b) Is this a convex optimization problem? Prove if it is convex; OR give an evidence that it is not convex.

c) Give the KKT conditions of your optimization problem.

d) Check if the height=1.5 meters, base-area = 6 square meters corresponds to an optimal solution.

Question 3 (10 points)

Derive the conjugate function $f^*(y)$ for each of the following functions:

a) $f(x) = 3x^2 + 4x$

b) $f(x) = -\log x + 2$

Question 4 (20 points)

Consider the function $f(\mathbf{x}) = f(x_1, x_2) = (x_1 + x_2)^2$.

a) Derive the gradient of $f(x)$.

b) At the point $\mathbf{x}_0 = (0, 1)^T$, consider the search direction $\mathbf{d} = (1, -1)^T$. Show that $\mathbf{d}$ is a descent direction.

c) Find the stepsize α that minimizes $f(\mathbf{x}_0 + \alpha\mathbf{d})$; that is, what is the result of this exact line search? Provide the value of $f(\mathbf{x}_0 + \alpha\mathbf{d})$.

d) Derive the Hessian of $f(x)$.

e) Perform one Newton step with $\alpha = 1$ starting at $\mathbf{x}_0 = (0, 1)^T$ to compute $\mathbf{x}_1$. What are $\mathbf{x}_1$ and $f(\mathbf{x}_1)$?

Question 5 (15 points)

The null space of a matrix $A \in \mathbb{R}^{m \times n}$ (with $m < n$) is defined to be the set of all $x \in \mathbb{R}^n$ such that $Ax = 0$. The projection of a point z onto a convex set $\mathcal{S}$ is defined as the point $x^* \in \mathcal{S}$ which is closest in Euclidean distance to z. Find a closed form solution for the projection of z onto the convex set $\{x \mid Ax = 0\}$. (You can assume A is full rank, i.e., rank $A = m$.) To solve the problem, set it up as a constrained optimization, write out the Lagrangian, and derive the KKT conditions.

Question 6 (10 points)

You are the manager of a large company where you face the decision of selecting the right projects to maximize the total returns. There are n possible projects $\{P_k\}_{k=1}^n$. Each project P_k runs for 3 years and has an overall return of c_k dollars. The financial constraints are that, in year t, there are only a total of f_t dollars available for these projects, whereas project P_k requires at least $a_{k,t}$ dollars (for $t = 1, 2, 3$ and $k = 1, \ldots, n$). Formulate this problem as an integer program (i.e., a problem with integer variables/constraints).

Hint: Define variables x_k so that $x_k = 1$ means selecting P_k and $x_k = 0$ means not selecting P_k. Formulate your integer program in terms of these variables.

Question 7 (15 points)

Consider the linear program

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 1,\ x_1 - x_2 \leq 1,\ x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

a) Draw the region of feasibility.

b) Solve the LP and verify the optimality of your optimal solution.

Solutions to Final Exam Winter 2017

Solutions to Question 1

a) Yes, the objective function and all the constraints are convex, and the equality constraint is linear.

b) $(0.5, 0.5)^T$ is an ideal Slater point because none of the inequality constraints are active.

c) The Lagrangian is

$$\begin{aligned} L(x,\lambda,\mu) =& f(x) + \sum_{j=1}^{k} \lambda_j g_j(x) + \mu^T(b - Ax) \\ =& e^{2x_1} + e^{-2x_2} + \lambda_1(x_1^2 + x_2^2 - 3x_1 - 4x_2) \\ & + \lambda_3(-x_1) + \lambda_4(-x_2) + \mu(x_1 + x_2 - 1)\,. \end{aligned}$$

Denote $g(\lambda,\mu) = \inf_x\{L(x,\lambda,\mu)\}$.

The Lagrange dual of this problem is

$$\begin{aligned} \sup_{\lambda,\mu} \quad & g(\lambda,\mu) \\ \text{s.t.} \quad & \lambda \geq 0\,. \end{aligned}$$

d) The KKT condition for this problem is

$$\begin{aligned} g(x) &\leq 0 \\ Ax &= b \\ x &\geq 0 \\ \lambda &\geq 0 \\ \nabla_x L(x,\lambda,\mu) &= 0 \\ \lambda_j g_j(x) &= 0, j = 1,\cdots,k\,, \end{aligned}$$

where

$$\nabla_x L(x,\lambda,\mu) = \begin{bmatrix} 2e^{2x_1} + \lambda_1(2x_1 - 3) - \lambda_3 + \mu \\ -2e^{-2x_2} + \lambda_1(2x_2 - 4) - \lambda_4 + \mu \end{bmatrix}.$$

Solutions to Question 2

a) The problem can be formulated as follows

$$\begin{aligned} \max_{l,d,h} \quad & |d \cdot l - h| \\ \text{s.t.} \quad & h \cdot d \cdot l \geq 9 \\ & d \cdot l \leq 6 \\ & h \geq 1 \\ & h \leq 2 \\ & l \geq 0 \\ & d \geq 0\,. \end{aligned}$$

b) Rewriting the first constraint in the standard form $-h \cdot d \cdot l + 9 \leq 0$ and computing the Hessian of the function $g_1(l,d,h) = -h \cdot d \cdot l + 9$, we get:

$$\nabla^2 g_1(l,d,h) = \begin{bmatrix} 0 & -h & -d \\ -h & 0 & -l \\ -d & -l & 0 \end{bmatrix}.$$

As the determinant of the Hessian is negative for $l, d, h \geq 0$, the function $g_1(l,d,h)$ is not convex. So, the first constraint is not convex and this is not a convex optimization problem.

c) Noting that $l \cdot d \geq 4.5 \geq h$, the Lagrange function of the optimization problem is:

$$L(l,d,h) = h - d \cdot l + y_1(9 - h \cdot d \cdot l) + y_2(d \cdot l - 6) + y_3(1 - h) + y_4(h - 2) - y_5 l - y_6 d,\, y \geq 0,\, y \in \mathcal{R}^6\,.$$

Consequently, the KKT conditions of the NLO formulation are:

$$
\begin{aligned}
9 - h \cdot d \cdot l &\leq 0 \\
d \cdot l - 6 &\leq 0 \\
1 - h &\leq 0 \\
h - 2 &\leq 0 \\
-l &\leq 0 \\
-d &\leq 0 \\
-d - y_1(h \cdot d) + y_2 d - y_5 &= 0 \\
-l - y_1(h \cdot l) + y_2 l - y_6 &= 0 \\
1 - y_1(l \cdot d) - y_3 + y_4 &= 0 \\
y_1(9 - h \cdot d \cdot l) &= 0 \\
y_2(d \cdot l - 6) &= 0 \\
y_3(1 - h) &= 0 \\
y_4(h - 2) &= 0 \\
y_5 l &= 0 \\
y_6 d &= 0 \\
y &\geq 0\,.
\end{aligned}
$$

d) If $h = 1.5$ and $l \cdot d = 6$, then the problem is feasible as all the constraints are satisfied. For $1 \leq h \leq 1.5$ the problem gets infeasible. For $1.5 < h \leq 2$ the objective function value is smaller than for $h = 1.5$. So, $h = 1.5$ and $l \cdot d = 6$ is the optimal solution.

Solutions to Question 3

a) $f^*(y) = \max_x x \cdot y - (3x^2 + 4x)$. By stationarity $y - 6x - 4 = 0 \Rightarrow x = \frac{y-4}{6}$. Thus $f^*(y) = \frac{(y-4)^2}{12}$.

b) $f^*(y) = \max_x x \cdot y + \ln(x) - 2$. If $y \geq 0$, then clearly$f^*(y) = \infty$. Otherwise, by stationarity $y + \frac{1}{x} = 0 \Rightarrow x = -\frac{1}{y}$. Thus $f^*(y) = -3 + \ln(-\frac{1}{y}) = -3 - \ln(-y)$.

Solutions to Question 4

1. $$\nabla f(x) = \begin{bmatrix} 2(x_1 + x_2) \\ 2(x_1 + x_2) \end{bmatrix}.$$

2. $\nabla f(x_0)^T d = 0$. Therefore, d is **not** a descent direction.

3. $f(x_0 + \alpha d) = 1$ for all α.

4. $$\nabla^2 f(x) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

5. $\nabla^2 f(x)$ is not invertible. No Newton steps are defined.

Solutions to Question 5

$\min_x \frac{1}{2}\|x - z\|_2^2$ such that $Ax = 0$. Lagrangian $L(x, \lambda) = \frac{1}{2}\|x - z\|_2^2 + \lambda^T Ax$. KKT conditions give $Ax^* = 0$ and $x^* - z + A^T\lambda^* = 0$. The second condition (on multiplying by A) yields $Az = AA^T\lambda^*$, implying $\lambda^* = (AA^T)^{-1}Az$, yielding $x^* = z - A^T(AA^T)^{-1}Az$.

Solutions to Question 6

The problem can be formulated as follows

$$\begin{aligned} \max_x \quad & \sum_k c_k \cdot x_k \\ \text{s.t.} \quad & \sum_k a_{kt} \cdot x_k \leq f_t, \quad t = 1, 2, 3 \\ & x_k \in \{0, 1\}, \quad k = 1, \cdots, n\,. \end{aligned}$$

Solutions to Question 7

a) Clear.

b) $(0, 0)^T$ is the optimal point, which can be verified by the Rockafellar-Pshenichnyi condition.