University of Waterloo
Department of Electrical and Computer Engineering
Winter, 2018

ECE 602: Introduction to Optimization

FINAL EXAMINATION

Surname								
Legal Given Name(s)								
UW Student ID Number								

Instruction:

1. There are 100 points in total.
2. This is a written, open-book exam (only unannotated lecture slides are allowed). Please turn off all electronic media and store them under your desk.
3. Be neat. Poor presentation will be penalized.
4. **No questions will be answered during the exam**. If in doubt, state your assumption(s) and continue.
5. Do not leave during the examination period without permission.
6. Do not stand up until all the exams have been picked up.

Do well!

Question 1 (20 points)

Show that for $p > 1$, the function

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{\|x\|_p^p}{t^{p-1}}$$

is convex on $\{(x,t) \mid t > 0\}$. (Note that the function maps from $\mathbb{R}^n \cup \mathbb{R}_{++}$ to $\mathbb{R}_+$.)

Question 2 (20 points)

Let $f(x,z)$ be convex in (x,z) and define $g(x) = \inf_z f(x,z)$. Express the conjugate g^* in terms of f^*. As an application, express the conjugate of $g(x) = \inf_z\{h(z) \mid Az + b = x\}$, where h is convex, in terms of h^*, A, and b.

Question 3 (20 points)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Formulate the following problems as LPs.

1. Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \le 1$.
2. Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \le 1$.
3. Minimize $\|Ax - b\|_1 + \|x\|_\infty$.

Question 4 (20 points)

Derive a dual problem for

$$\text{minimize}_x \ \sum_{i=1}^{N} \|A_i x + b_i\|_2 + (1/2)\|x - x_0\|_2^2,$$

where $A_i \in \mathbb{R}^{m_i \times n}$, $b_i \in \mathbb{R}^{m_i}$, and $x_0 \in \mathbb{R}^n$. (Hint: First introduce new variables $y_i \in \mathbb{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.)

Question 5 (20 points)

Formulate the following problem as a convex minimization problem. Find the rectangle

$$\mathcal{R} = \{x \in \mathbb{R}^n \mid u \preceq x \preceq l\}$$

of maximum volume, enclosed in a polyhedron $\mathcal{P} = \{x \mid Ax \preceq b\}$. The variables are u and l.

Solutions to Final Exam
Winter, 2018

Question 1

Show that for $p > 1$,

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{\|x\|_p^p}{t^{p-1}}$$

is convex on $\{(x,t) \mid t > 0\}$.
Solution. This is the perspective function of $\|x\|_p^p = |x_1|^p + \cdots + |x_n|^p$.

Question 2

Conjugate and minimization. Let $f(x,z)$ be convex in (x,z) and define $g(x) = \inf_z f(x,z)$. Express the conjugate g^* in terms of f^*.
As an application, express the conjugate of $g(x) = \inf_z\{h(z) \mid Az + b = x\}$, where h is convex, in terms of h^*, A, and b.
Solution.

$$\begin{aligned} g^*(y) &= \sup_x (x^T y - \inf_z f(x,z)) \\ &= \sup_{x,z} (x^T y - f(x,z)) \\ &= f^*(y,0). \end{aligned}$$

To answer the second part of the problem, we apply the previous result to

$$f(x,z) = \begin{cases} h(z) & Az + b = x \\ \infty & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} f^*(y,v) &= \inf(y^T x - v^T z - f(x,z)) \\ &= \inf_{Az+b=x} (y^T x - v^T z - h(z)) \\ &= \inf_z (y^T(Az+b) - v^T z - h(z)) \\ &= b^T y + \inf_z (y^T Az - v^T z - h(z)) \\ &= b^T y + h^*(A^T y - v). \end{aligned}$$

Therefore

$$g^*(y) = f^*(y,0) = b^T y + h^*(A^T y).$$

Question 3

a)

Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -\mathbf{1} \leq x \leq \mathbf{1}, \end{array}$$

with variables $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$.

b)

Equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \leq x \leq y \\ & -\mathbf{1} \leq Ax - b \leq \mathbf{1} \end{array}$$

with variables x and y.

c)

Equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y + t \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -t\mathbf{1} \preceq x \preceq t\mathbf{1}, \end{array}$$

with variables x, y, and t.

Question 4

Derive a dual problem for

$$\text{minimize} \quad \sum_{i=1}^N \|A_i x + b_i\|_2 + (1/2)\|x - x_0\|_2^2.$$

The problem data are $A_i \in \mathbf{R}^{m_i \times n}$, $b_i \in \mathbf{R}^{m_i}$, and $x_0 \in \mathbf{R}^n$. First introduce new variables $y_i \in \mathbf{R}^{m_i}$ and equality constraints $y_i = A_i x + b_i$.

Solution. The Lagrangian is

$$L(x, z_1, \ldots, z_N) = \sum_{i=1}^N \|y_i\|_2 + \frac{1}{2}\|x - x_0\|_2^2 - \sum_{i=1}^N z_i^T (y_i - A_i x - b_i).$$

We first minimize over y_i. We have

$$\inf_{y_i}(\|y_i\|_2 + z_i^T y_i) = \begin{cases} 0 & \|z_i\|_2 \le 1 \\ -\infty & \text{otherwise.} \end{cases}$$

(If $\|z_i\|_2 > 1$, choose $y_i = -tz_i$ and let $t \to \infty$, to show that the function is unbounded below. If $\|z_i\|_2 \le 1$, it follows from the Cauchy-Schwarz inequality that $\|y_i\|_2 + z_i^T y_i \ge 0$, so the minimum is reached when $y_i = 0$.)

We can minimize over x by setting the gradient with respect to x equal to zero. This yields

$$x = x_0 + \sum_{i=1}^N A_i^T z.$$

Substituting in the Lagrangian gives the dual function

$$g(z_1, \ldots, z_N) = \begin{cases} \sum_{i=1}^N (A_i x_0 + b_i)^T z_i - \frac{1}{2}\|\sum_{i=1}^N A_i^T z_i\|_2^2 & \|z_i\|_2 \le 1, \ i = 1, \ldots, N \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem is

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^N (A_i x_0 + b_i)^T z_i - \frac{1}{2}\|\sum_{i=1}^N A_i^T z_i\|^2 \\ \text{subject to} & \|z_i\|_2 \le 1, \ i = 1, \ldots, N. \end{array}$$

Question 5

Maximum volume rectangle inside a polyhedron. Formulate the following problem as a convex optimization problem. Find the rectangle

$$\mathcal{R} = \{x \in \mathbf{R}^n \mid l \preceq x \preceq u\}$$

of maximum volume, enclosed in a polyhedron $\mathcal{P} = \{x \mid Ax \preceq b\}$. The variables are $l, u \in \mathbf{R}^n$. Your formulation should not involve an exponential number of constraints.

Solution. A straightforward, but very inefficient, way to express the constraint $\mathcal{R} \subseteq \mathcal{P}$ is to use the set of $m2^n$ inequalities $Av^i \preceq b$, where v^i are the (2^n) corners of $\mathcal{R}$. (If the corners of a box lie inside a polyhedron, then the box does.) Fortunately it is possible to express the constraint in a far more efficient way. Define

$$a_{ij}^+ = \max\{a_{ij}, 0\}, \qquad a_{ij}^- = \max\{-a_{ij}, 0\}.$$

Then we have $\mathcal{R} \subseteq \mathcal{P}$ if and only if

$$\sum_{i=1}^{n} (a_{ij}^+ u_j - a_{ij}^- l_j) \leq b_i, \quad i = 1, \ldots, m,$$

The maximum volume rectangle is the solution of

$$\begin{array}{ll} \text{maximize} & \left(\prod_{i=1}^n (u_i - l_i)\right)^{1/n} \\ \text{subject to} & \sum_{i=1}^n (a_{ij}^+ u_j - a_{ij}^- l_j) \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

with implicit constraint $u \succeq l$.