University of Waterloo
Department of Electrical and Computer Engineering
Winter, 2022

ECE 602: Introduction to Optimization

FINAL EXAMINATION

Surname								
Legal Given Name(s)								
UW Student ID Number								

<u>Instruction:</u>

1. There are 100 points in total.
2. This is a written, open-book exam. Please turn off all electronic media and store them under your desk.
3. Be neat. Poor presentation will be penalized.
4. **No questions will be answered during the exam**. If in doubt, state your assumption(s) and proceed.
5. Do not leave during the examination period without permission.
6. Do not stand up until all the exams have been picked up.

Do well!

Question 1 (20 points)

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and convex, and $g : \mathbb{R}^n \to \mathbb{R}$ is positive and concave. Show that the function $h(x) = f(x)^2/g(x)$, with domain $\mathbf{dom}\, h = \mathbf{dom}\, f \cap \mathbf{dom}\, g$, is convex.

Question 2 (20 points)

Recall that the *infimal convolution* of two functions f and g on $\mathbb{R}^n$ is defined as

$$h(x) = \inf_y \left(f(y) + g(x-y)\right).$$

For $f(x) = \|x\|_1$ and $g(x) = (1/2)\|x\|_2^2$, show that

$$h(x) = \inf_y \left(f(y) + g(x-y)\right) = \inf_y \left(\|y\|_1 + (1/2)\|x-y\|_2^2\right)$$

is the Huber penalty

$$h(x) = \sum_{i=1}^n \varphi(x_i), \quad \text{with} \quad \varphi(u) = \begin{cases} u^2/2, & |u| \le 1 \\ |u| - 1/2, & |u| > 1 \end{cases}.$$

Question 3 (20 points)

Consider a random optimization problem of the form

$$\begin{aligned} \min_{x,\beta} \ & \beta \\ \text{subject to } & \mathbf{prob}(c^T x \ge \beta) \le \alpha \\ & Ax = b \end{aligned}$$

Is this problem a convex optimization problem? If yes, rewrite the problem in a standard form.

Question 4 (20 points)

Consider the problem of projecting a point $a \in \mathbb{R}^n$ on the unit ball in ℓ_1-norm:

$$\begin{aligned} \min_x \ & (1/2)\|x - a\|_2^2 \\ \text{subject to } & \|x\|_1 \le 1. \end{aligned}$$

Derive the dual problem and describe an efficient method for solving it. Explain how you can obtain the optimal x from the solution of the dual problem.

Question 5 (20 points)

Suppose $a \in \mathbb{R}^n$ with $a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n > 0$, and $b \in \mathbb{R}^n$ with $b_k = 1/a_k$. Derive the KKT conditions for the convex optimization problem

$$\begin{aligned} \min_x \ & -\log(a^T x) - \log(b^T x) \\ \text{subject to } & x \succeq 0, \quad \mathbf{1}^T x = 1. \end{aligned}$$

Show that $x = [1/2, 0, 0, \ldots, 0, 0, 1/2]^T$ is optimal.