University of Waterloo
Department of Electrical and Computer Engineering
Spring, 2023

ECE 602: Introduction to Optimization

FINAL EXAMINATION

Surname								
Legal Given Name(s)								
UW Student ID Number								

Instruction:

1. There are 100 points in total.
2. This is a written, open-book exam. Please turn off all electronic media and store them under your desk.
3. Be neat. Poor presentation will be penalized.
4. **No questions will be answered during the exam**. If in doubt, state your assumption(s) and proceed.
5. Do not leave during the examination period without permission.
6. Do not stand up until all the exams have been picked up.

Do well!

Question 1 (25 points)

Formulate the following problems as LPs.

a) $\min \|Ax - b\|_1$ subject to $\|x\|_\infty \le 1$.

b) $\min \|x\|_1$ subject to $\|Ax - b\|_\infty \le 1$.

c) $\min \|Ax - b\|_1 + \|x\|_\infty$.

In each problem, $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ are given.

Question 2 (25 points)

Derive a Lagrange dual for the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} \phi(r_i) \\ \text{subject to} & r = Ax - b, \end{array}$$

where

$$\phi(u) = \begin{cases} u^2, & |u| \le 1 \\ 2|u| - 1, & |u| > 1. \end{cases}$$

Question 3 (25 points)

Consider the function $f(\mathbf{x}) = f(x_1, x_2) = (x_1 + x_2)^2$.

a) Derive the gradient of $f(x)$.

b) At the point $\mathbf{x}_0 = (0, 1)^T$, consider the search direction $\mathbf{d} = (1, -1)^T$. Show that $\mathbf{d}$ is a descent direction.

c) Find the stepsize α that minimizes $f(\mathbf{x}_0 + \alpha\mathbf{d})$; that is, what is the result of this exact line search? Provide the value of $f(\mathbf{x}_0 + \alpha\mathbf{d})$.

d) Derive the Hessian of $f(x)$.

e) Perform one Newton step with $\alpha = 1$ starting at $\mathbf{x}_0 = (0, 1)^T$ to compute $\mathbf{x}_1$. What are $\mathbf{x}_1$ and $f(\mathbf{x}_1)$?

Question 4 (25 points)

Formulate the following problem as a convex minimization problem. Find the rectangle

$$\mathcal{R} = \{x \in \mathbb{R}^n \mid u \preceq x \preceq l\}$$

of maximum volume, enclosed in a polyhedron $\mathcal{P} = \{x \mid Ax \preceq b\}$. The variables are u and l.

Solutions

Question 1

a) With $t \in \mathbf{R}^m$, the equivalent problem is given by

$$\begin{aligned} \min_{x,t} \quad & \mathbf{1}^T t \\ \text{s.t.} \quad & -t \preceq Ax - b \preceq t \\ & -\mathbf{1} \preceq x \preceq \mathbf{1} \end{aligned}$$

with implicit constraint $t \succeq 0$.

b) With $t \in \mathbf{R}^m$, the equivalent problem is given by

$$\begin{aligned} \min_{x,t} \quad & \mathbf{1}^T t \\ \text{s.t.} \quad & -t \preceq x \preceq t \\ & -\mathbf{1} \preceq Ax - b \preceq \mathbf{1} \end{aligned}$$

with implicit constraint $t \succeq 0$.

c) With $t \in \mathbf{R}^m$ and $s \in \mathbf{R}$, the equivalent problem is given by

$$\begin{aligned} \min_{x,t,s} \quad & \mathbf{1}^T t + s \\ \text{s.t.} \quad & -t \preceq Ax - b \preceq t \\ & -s\,\mathbf{1} \preceq x \preceq s\,\mathbf{1} \end{aligned}$$

with implicit constraints $t \succeq 0$ and $s > 0$.

Question 2

The Lagrangian is given by

$$L(x, r, \nu) = \sum_{i=1}^{m} \phi(r_i) + \nu^T(Ax - b - r) = \sum_{i=1}^{m} \phi(r_i) + (A^T\nu)^T x - \nu^T(b + r).$$

Note that, as a function of x, $L(x, r, \nu)$ is unbounded below unless $A^T\nu = 0$. When the equality holds, we have

$$g(\nu) = \inf_x L(x, r, \nu) = -\nu^T b + \inf_r \left(\sum_{i=1}^{m} \phi(r_i) - \nu^T r \right) = -\nu^T b - \sum_{i=1}^{m} \sup_{r_i}(-\phi(r_i) + \nu_i r_i) =$$

$$= -\nu^T b - \sum_{i=1}^{m} \phi^*(\nu_i),$$

where ϕ^* is the convex conjugate of ϕ. Consequently, the dual problem becomes

$$\max_{\nu} \; -\nu^T b - \sum_{i=1}^{m} \phi^*(\nu_i), \quad \text{s.t. } A^T\nu = 0.$$

Finally, the convex conjugate of the Huber penalty can be shown to be

$$\phi^*(y) = \begin{cases} y^2/4, & \text{if } |y| \leq 2 \\ \infty, & \text{otherwise} \end{cases}$$

which results in a QP given by

$$\max_{\nu} \; -\nu^T b - \|\nu\|_2^2/4, \quad \text{s.t. } A^T\nu = 0, \; \|\nu\|_\infty \leq 2.$$

Question 3

a) The gradient of f is equal to $\nabla f(x) = (2(x_1 + x_2),\, 2(x_1 + x_2))^T$.

b) At point $x_0 = [0, 1]^T$, the gradient is equal to $\nabla f(x_0) = (2,\; 2)^T$. For the direction $d = (1, -1)^T$ to be a strictly descend direction, we must have $\nabla f(x_0)^T d < 0$. Instead, we have

$$\nabla f(x_0)^T d = [2,\; 2][1,\; -1]^T = 0.$$

Thus, d is not a strictly descend direction.

c) For an arbitrary $\alpha \geq 0$, we have

$$x = x_0 + \alpha\, d = (0,\; 1)^T + \alpha\, (1,\; -1)^T = (\alpha,\; 1 - \alpha)^T.$$

Consequently,

$$f(x_0 + \alpha\, d) = (\alpha + 1 - \alpha)^2 = 1.$$

The function remains constant along the chosen direction.

d) The Hessian of f is given by

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

e) Since the Hessian is not invertible, the Newton method cannot be applied in this case.

Question 4

Define $a_{ij}^+ = \max\{a_{ij}, 0\}$ and $a_{ij}^- = \max\{-a_{ij}, 0\}$. Then, $\mathcal{R} \subseteq \mathcal{P}$ if and only if

$$\sum_{i=1}^{n} (a_{ij}^+ u_j - a_{ij}^- l_j) \leq b_i, \quad i = 1, 2, \ldots, m.$$

Therefore, the maximum volume rectangle is the solution of

$$\begin{aligned} \max_{u,l} \; & \left(\prod_{i=1}^{n} (u_i - l_i) \right)^{1/n} \\ \text{s.t. } & \sum_{i=1}^{n} (a_{ij}^+ u_j - a_{ij}^- l_j) \leq b_i, \quad i = 1, 2, \ldots, m. \end{aligned}$$

with implicit constraint $u \succeq l$.