University of Waterloo
Department of Electrical and Computer Engineering
Winter, 2024

ECE 602: Introduction to Optimization

FINAL EXAMINATION

Surname								
Legal Given Name(s)								
UW Student ID Number								

Instruction:

1. There are 100 points in total.
2. This is a written, open-book exam. Please turn off all electronic media and store them under your desk.
3. Be neat. Poor presentation will be penalized.
4. **No questions will be answered during the exam**. If in doubt, state your assumption(s) and proceed.
5. Do not leave during the examination period without permission.
6. Do not stand up until all the exams have been picked up.

Do well!

Question 1 (25 points)

In class, we formulated the problem of finding the largest Euclidean ball that lies in a polyhedron

$$\mathcal{P} = \{x \in \mathbf{R}^n \mid a_i^T x \leq b_i,\ i = 1, \cdots, m\}$$

as a linear program. What happens if, in this problem, we replace the largest Euclidean ball with the largest ball in the l_∞-norm, i.e., $\mathcal{B} = \{x_c + ru \mid \|u\|_\infty \leq 1\}$? Can the resulting problem still be formulated as an LP? If yes, show how.

Question 2 (25 points)

Consider the optimization problem

$$\begin{aligned} \min\ & f_0(x_1, x_2) \\ \text{s.t.}\ & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Make a sketch of the feasible set. For each of the following objective functions, provide the optimal solution $x^* = (x_1^*, x_2^*)$ and the optimal value p^*.

a) $f_0(x_1, x_2) = x_1 + x_2$.

b) $f_0(x_1, x_2) = -x_1 - x_2$.

c) $f_0(x_1, x_2) = x_1$.

Question 3 (25 points)

Consider the following optimization problem with *one* inequality constraint

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & f(x) \leq 0, \end{aligned}$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is not necessarily convex. Express the dual of this problem in terms of the convex conjugate f^* of f. Explain why the dual problem is convex.

Question 4 (25 points)

Consider the following problem

$$\begin{aligned} \min_x\ & x^T P x \\ \text{s.t.}\ & ||x||_2 = 1 \end{aligned}$$

where $x \in \mathbf{R}^n$ and $P \in \mathbf{S}^n$.

a) Is it a convex optimization problem? Explain your answer.

b) What are the optimal value $p^\star$ of the problem and the corresponding optimal solution $x^\star$? How do $p^\star$ and $x^\star$ depend on P?

Solutions

Question 1

Yes, the new problem is actually an LP. To see that, we note that, for $\mathcal{B} \subseteq \mathcal{P}$, we need to have

$$\sup_{\|u\|_\infty \leq 1} \left(a_i^T(x_c + r\,u)\right) = a_i^T x_c + r \sup_{\|u\|_\infty \leq 1} a_i^T u = a_i^T x_c + r\|a_i\|_1 \leq b_i, \quad i = 1, 2, \ldots, m$$

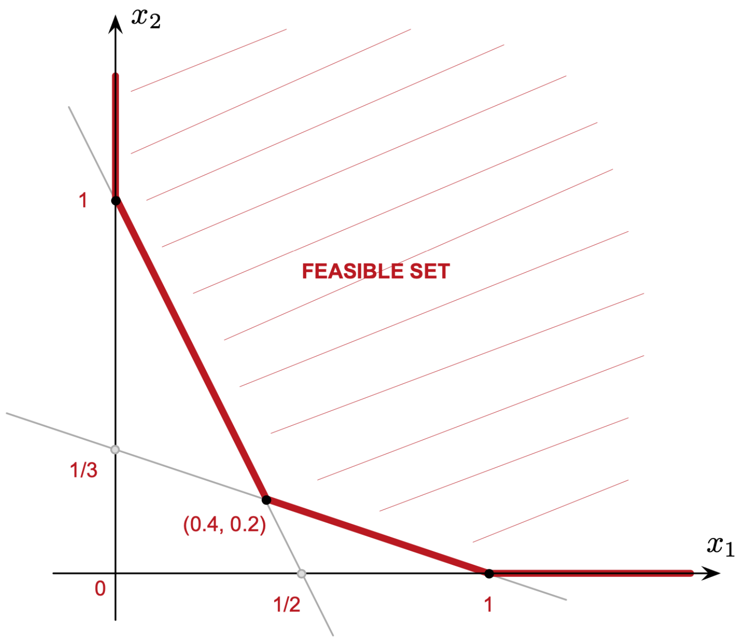
Therefore, the resulting problem is given by

$$\begin{aligned} &\max_{x_c, r \geq 0} \; r \\ &\text{s.t.} \;\; a_i^T x_c + r\|a_i\|_1 \leq b_i, \quad i = 1, 2, \ldots, m \end{aligned}$$

which is an LP.

Question 2

The feasible set is shown in the figure below.



a) In this case, the linear cost increases in the direction of vector $(1,\ 1)^T$. As a result, $x^* = (0.4, 0.2)$ and $p^* = 0.4 + 0.2 = 0.6$.

b) In this case, the linear cost decreases in the direction of vector $(1,\ 1)^T$, in which case $p^* = -\infty$. There are no optimal solutions.

c) In this case, we have $p^* = 0$, since $x_1 \geq 0$. There are infinitely many optimal solutions of the form $x^* = (0, \alpha)$, with $\alpha \geq 1$.

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Question 3

The Lagrangian of the problem is given by

$$L(x,\lambda) = c^T x + \lambda f(x).$$

Consequently,

$$g(\lambda) = \inf_x \left(c^T x + \lambda f(x)\right) = -\lambda \sup_x \left((-c/\lambda)^T x - f(x)\right) = -\lambda f^*(-c/\lambda).$$

In this case, the dual problem has the form

$$\begin{aligned} \max_\lambda \ & -\lambda f^*(-c/\lambda) \\ \text{s.t.} \ & \lambda \geq 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} \min_\lambda \ & \lambda f^*(-c/\lambda) \\ \text{s.t.} \ & \lambda \geq 0 \end{aligned}$$

Note that f^* is convex, and therefore its composition with the perspective function is convex in λ, for all $\lambda > 0$ (see Slide 31, Section 2). Consequently, the dual optimization problem is convex.

Question 4

The given problem is equivalent to: $\min_x \ x^T P\, x$, s.t. $x^T x = 1$.

a) Since P is not positive-definite and the equality constraint is nonlinear, the given problem is not a convex optimization problem.

b) The Lagrangian is given by

$$L(x,\nu) = x^T P\, x + \nu\,(1 - x^T x).$$

In this case, the 1st-order optimality requires

$$\nabla_x L(x^*,\nu^*) = 2Px^* - 2\nu^* x^* = 0$$

resulting in

$$Px^* = \nu^* x^*$$

Thus, the primal optimal x^* is an eigenvector of P, while the dual optimal ν^* is equal to the eigenvalue of P corresponding to this eigenvector. Moreover,

$$p^* = (x^*)^T P\, x^* = \nu^*\,(x^*)^T x^* = \nu^* \|x^*\|_2^2 = \nu^*.$$

Therefore, the optimal value p^* is defined by the minimal value of ν^* which, in turn, is equal to the smallest eigenvalue $\lambda_{\min}(P)$ of P. In summary, $p^* = \lambda_{\min}(P)$, while x^* is given by the eigenvector of P corresponding to $\lambda_{\min}(P)$.