Abstract—We study the problem of strategic network formation among a set of nodes where each node forms links with other nodes in the network to maximize some utility. While previous work in this area has considered the formation of a single edge set between the nodes, we consider the problem of the strategic formation of multiple edge sets between the nodes, corresponding to different types of relationships. We start by considering the case where one edge set is chosen to minimize distances between nodes that are neighbors in another edge set. This corresponds to a generalization of distance-based utility functions studied in the literature. In this setting, we characterize efficient networks (that are optimal with respect to a global utility function), and pairwise stable networks (where individual nodes cannot benefit from the addition or removal of incident edges). We then generalize existing concepts of pairwise stability and improving paths for network formation to the multi-layer setting with arbitrary utility functions.

I. INTRODUCTION

Examples of complex networks abound in both the natural world (e.g., ecological, social and economic systems), and in engineered applications (e.g., the Internet, the power grid, etc.). The topological structure of such networks (i.e., the relationships and interactions between the various nodes) plays a fundamental role in the functioning of the network. Early research on the structure of complex networks primarily adopted a stochastic perspective, postulating that the links between nodes are formed randomly [1], [2]. An alternative perspective, driven by the economics, computer science and engineering communities, has argued that optimization (rather than pure randomness) must play a key role in network formation. In such settings, each node in the network forms links with other nodes in order to optimize local utility functions, resulting in networks that can be analyzed using game-theoretic notions of equilibria and efficiency. One can also consider a central designer that determines the optimal connections in the network to maximize the overall utility.

The last decade has generated a large volume of literature on the problem of network formation for different types of utility functions [3]–[6]. In [7], [8], Jackson and Wolinsky introduced the distance-based utility function for network formation, where the objective is to purchase edges to minimize the distance between any two nodes in the network. A characterization of the price of anarchy (i.e., the worst case loss in optimality due to selfish local optimization as compared to the global optimum) for network formation with distance-based utility functions was given in [9].

A common theme in the existing works on strategic network formation is that they focus on the construction of a single set of edges between the nodes. However, many real-world networks inherently consist of multiple layers of relationships between the same set of nodes. Examples include friendship and professional relationships in social networks, policy influence and knowledge exchange in organizational networks [10], and coupled communication and energy infrastructure networks [11]. While there has been growing research on different aspects of multi-layer networks in recent years [10], [12]–[14], the problem of strategic multi-layer network formation has not been addressed yet.

We begin an investigation of this problem in this paper by generalizing distance-utility network formation to the case where one layer (or graph) is formed by optimizing the distances between nodes who have relationships in another layer (or graph). As a motivating example, consider the problem in [15], where both the physical infrastructure network and the traffic flow network between a group of cities are studied. Traffic flow can be interpreted as the importance or weight of the connection between the endpoint cities and thus an important objective is to design an optimal infrastructure network between cities with respect to the given traffic flow pattern. In the simplest case, this problem can be modeled as a network formation problem with a distance-based utility function where only the distance between specific pairs of nodes matters (i.e., the pairs with sufficiently high traffic flow between them). We address this class of problems in this paper by first defining a graph (or layer) $G_1$ capturing an existing set of relationships between nodes, and then studying the formation of a second graph $G_2$ based on $G_1$.

We then build upon these results to investigate the problem of designing multiple networks simultaneously based on local objective functions. Unlike the first part of this paper where a central designer chooses the optimal network, in the second part we consider the situation where each node decides about its edges to other nodes in the network. We assume general utility functions for the nodes and introduce the concept of intra-layer pairwise stable networks, where no node in a given multi-layer network can improve its utility via the addition or deletion of a single edge in any layer. Using the idea of improving paths in single-layer network formation [16], we prove that certain dynamics for multi-layer network formation either lead to an intra-layer pairwise stable network or cycle forever through a fixed set of networks. Finally, we propose a family of utility functions for which the multi-layer network formation problem can be reduced to a set of single-layer network formation problems.
II. DEFINITIONS

An undirected network (or graph) is denoted as \( G = (N, E) \) where \( N = \{1, 2, \ldots, n\} \) is the set of nodes (or vertices) and \( E \subseteq \{(i, j) | i, j \in N, i \neq j\} \). Two nodes are said to be connected if there is an edge between them. A path from node \( v_1 \) to \( v_k \) in graph \( G \) is a sequence of distinct nodes \( v_1 v_2 \ldots v_k \) where there is an edge between any consecutive nodes of the sequence. The length of a path is the number of edges in the sequence. The shortest path between nodes \( i \) and \( j \) in graph \( G \) is denoted by \( d_G(i, j) \). A cycle is a path of length two or more from a node to itself. A graph \( G' = (N', E') \) is called a subgraph of \( G = (N, E) \), denoted as \( G' \subseteq G \), if \( N' \subseteq N \) and \( E' \subseteq E \cap \{N' \times N'\} \). A graph \( G' \) is said to be induced by a set of nodes \( N' \subseteq N \) if \( E' = E \cap \{N' \times N'\} \). A graph is connected if there is a path from every node to every other node. A tree is a connected acyclic graph. A subgraph \( G' \subseteq G \) is a component of \( G \) if nodes in \( G' \) and \( G \setminus G' \) are not connected and \( G' \) is a connected subgraph induced by its vertex set. A forest is a disconnected graph such that its components are trees.

III. DISTANCE-BASED UTILITY

A canonical problem in network formation introduced by Jackson and Wolinsky involves distance-based utilities [7]. In this model, there is a net benefit of \( b(k) \) to the central designer for each pair of nodes that are \( k \) hops away from each other in the network, where \( b(\cdot) \) is a decreasing nonnegative function with \( b(\infty) = 0 \) (i.e., nodes that are further away provide smaller benefits). There is a cost \( c > 0 \) for each edge in the network. Let \( N = \{1, 2, \ldots, n\} \) denote the set of nodes and \( G^N \) be the set of all graphs on \( N \). The outcome of the network formation process is a graph \( G = (N, E) \in G^N \). A graph \( G \) has an associated utility function (or value function) \( u: G^N \rightarrow \mathbb{R} \) given by

\[
    u(G) = \sum_{i, j \in N \mid i \neq j} b(d_G(i, j)) - c|E|
\]

In this formulation, there is an inherent trade-off faced by the designer: adding links to a larger number of nodes yields a larger benefit (by reducing the distances between nodes), but also a larger cost invested in links. An efficient network is the network with the highest utility. In other words, if \( G \) is an efficient network, then

\[
    u(G) \geq u(G'), \quad \forall G' \in G^N.
\]

A representative result in this setting is that there are only a few different kinds of efficient networks, depending on the relative values of the link costs and connection benefits: the empty network (for high link costs), the star network (for medium link costs), and the fully connected network (for low link costs) [7], [8].

IV. TWO LAYER DISTANCE-BASED UTILITIES: BEST RESPONSE GRAPH

We start by generalizing the study of distance-based network formation to a multi-layer setting. Specifically, suppose that we have a layer (or graph) \( G_1 = (N, E_1) \), where the edge set \( E_1 \) specifies a type of relationship between the nodes in \( N \). Our objective is to design another layer (or graph) \( G = (N, E) \) on the same set of nodes, where the utility of the graph is given by

\[
    u(G|G_1) = \sum_{(i, j) \in E_1} b(d_G(i, j)) - c|E|.
\]

This utility function captures the idea that only distances between certain pairs of nodes (specified by the edge set \( E_1 \)) matter in the graph \( G \), as opposed to the distances between all pairs of nodes, as in the traditional distance-based network formation model described in equation (1). Indeed the traditional distance utility function in (1) is obtained when \( G_1 \) is the complete graph. As an example, consider each edge in \( E_1 \) as a pair of nodes that need to communicate (perhaps in a multi-hop fashion) over the physical communication network captured by edge set \( E \). The problem of optimally designing \( E \) with respect to \( E_1 \) can be cast as maximizing the utility function (2). Assume \( G_2 = (N, E_2) \) is the network that maximizes (2); we say \( G_2 \) is the best response network to \( G_1 \) or, equivalently, the efficient network with respect to the utility function (2). A natural conjecture is that the best response network with respect to a network \( G_1 \) is always a subgraph of \( G_1 \). This is trivially true when \( G_1 \) is the complete network, but the following example demonstrates that this is not necessarily the case when \( G_1 \) is a general graph.

Example 1: Consider graphs \( G_1 \) and \( G_2 \) shown in Figure 1. Let \( c = 1 \) and \( b(k) = \delta^k \) where \( \delta = 0.87 \). One can check that \( u(G_2|G_1) = 0.1955, u(G_1|G_1) = -1.17 \), and \( u(G_3|G_1) = -0.5416 \) where \( G_3 \) is any subgraph of \( G_1 \) with 8 edges. Furthermore, \( u(G_4|G_1) \leq 0.0868 \) for any network \( G_4 \) obtained by removing two or more arbitrary edges of \( G_1 \). As \( G_2 \) has a higher utility than any subgraph of \( G_1 \), we see that the best-response to \( G_1 \) cannot be a subgraph of \( G_1 \). In this example, one can verify (e.g., using a brute-force search) that \( G_2 \) is in fact the best response network to \( G_1 \).

\[ \square \]

![Fig. 1: Network \( G_1 \) (left) and corresponding best response \( G_2 \) (right).](image)

We will now characterize certain properties of best response networks. We start with the following useful result.

**Lemma 1:** If \( G_2 \) is the best response network to \( G_1 \), then the number of edges in \( G_2 \) is less than or equal to the number of edges in \( G_1 \), with equality if and only if \( G_2 = G_1 \).

**Proof:** We use contradiction to prove the first part. Suppose that \( G_2 \) is the best response and has more edges
than $G_1$. Then
\[
u(G_2|G_1) = \sum_{(u,v) \in E_1} b(d_{G_2}(u,v) - c)|E_2|\]
\[\leq |E_1|b(1) - c|E_2|\]
\[< |E_1|b(1) - c|E_1|\]
\[= \nu(G_1|G_1),\]
which contradicts our assumption that $G_2$ is the best response to $G_1$. To prove the second part, note that if $G_2 = G_1$ then the number of edges in $G_2$ and $G_1$ are equal. So we only need to show that if the number of edges in $G_2$ is equal to the number of edges in $G_1$ and $G_2$ is the best response to $G_1$, then $G_2 = G_1$. If $G_2 \neq G_1$ then
\[
u(G_2|G_1) = \sum_{(u,v) \in E_1} b(d_{G_2}(u,v) - c)|E_2|\]
\[< |E_1|b(1) - c|E_1|\]
\[= \nu(G_1|G_1),\]
which is again not possible since we assumed that $G_2$ is the best response to $G_1$.

The next lemma discusses the best response network with respect to subgraphs of $G_1$.

Lemma 2: Let $G_2$ be the best response network to $G_1$, and suppose that $G_2$ is not connected. Let $G_{2i} = (N_i, E_{2i})$, $i = 1, \ldots, k$, be the components of $G_2$. Let $G_{1i} = (N_i, E_{1i})$, $i = 1, 2, \ldots, k$, be the subgraphs induced by vertex sets $N_i$ on $G_1$. Then network $G_{2i}$ must be the best response network to $G_{1i}$ for $i = 1, 2, \ldots, k$.

Proof: Consider the utility of network $G_2$ with respect to $G_1$. Since there are no edges between the components in $G_2$, for any $(u, v) \in E_1$ with $u$ and $v$ in different components of $G_2$, $d_{G_2}(u,v) = \infty$. Thus \[\sum_{(u,v) \in E_1} b(d_{G_2}(u,v)) = \sum_{i=1}^{k} \sum_{(u,v) \in E_{1i}} b(d_{G_2}(u,v)),\] and the utility function can be written as
\[
u(G_2|G_1) = \sum_{(u,v) \in E_1} b(d_{G_2}(u,v) - c)|E_2|\]
\[= \sum_{i=1}^{k} \left( \sum_{(u,v) \in E_{1i}} b(d_{G_2}(u,v) - c)|E_{2i}| \right)\]
\[= \nu(G_{21}|G_{11}) + \cdots + \nu(G_{2k}|G_{1k}).\]

Now, if $G_{2i}$ is not the best response to $G_{1i}$ for some $i \in \{1, 2, \ldots, k\}$, replace it with the best response. This will increase the utility function, contradicting the fact that $G_2$ is the best response.

In general, the solution to the optimization problem (2) depends on both $G_1$ and the relative sizes of the benefit function and edge cost. In the following, we will characterize the best response network in certain cases.

Proposition 1: Suppose $G_1 = (N, E_1)$ is connected. If $b(1) > c$, then the best response network to $G_1$ is also connected.

Proof: Assume the best response network $G_2 = (N, E_2)$ is not connected. Then $\exists (u, v) \in E_1$ such that $d_{G_2}(u, v) = \infty$. For $G'_2 = (N, E'_2)$ with $E'_2 = E_2 \cup \{(u, v)\}$, \[\nu(G'_2|G_1) - \nu(G_2|G_1) \geq b(1) - c > 0.\]

This contradicts the assumption that $G_2$ is the best response network to $G_1$.

The next two propositions describe the best response network with respect to a general network in two specific cases.

Proposition 2: Let $G_1$ be an arbitrary graph. If $b(1) - c > b(2)$, then the best response network to $G_1$ is $G_2 = G_1$.

Proof: Suppose that $G_2$ is the best response network and $G_2 \neq G_1$. By Lemma 1, we know that the number of edges in $G_2$ is less than $G_1$. So there are vertices $u$ and $v$ such that $(u, v) \in E_1$ and $d_{G_2}(u, v) > 1$. Adding the edge $(u, v)$ to $E_2$ increases the utility by at least $b(1) - c - b(2) > 0$ which contradicts the assumption that $G_2 \neq G_1$ is the best response network. Therefore, the best response network must be equal to $G_1$.

Proposition 3: If $b(1) < c$, then the best response network to network $G_1$ is not $G_1$, unless $G_1$ is the empty network.

Proof: If $G_2 = G_1 \neq \phi$, then $\nu(G_2|G_1) = |E_1|(b(1) - c) < 0$ due to the assumption that $b(1) < c$. Thus it must be the case that $G_2 \neq G_1$, or $G_1$ is the empty network.

The above results lead to the following complete characterizations of the best response network when $G_1$ is a tree or a forest.

Corollary 1: Suppose $G_1 = (N, E_1)$ is a tree. Then the best response network $G_2 = (N, E_2)$ with respect to $G_1$ is $G_2 = G_1$ if $b(1) > c$. Otherwise if $b(1) < c$, $G_2$ is the empty graph.

Proof: For $b(1) > c$, by Proposition 1, $G_2$ must be a connected graph, and thus has at least $|N| - 1$ edges. By Lemma 1 and the fact that $G_1$ is a tree and has $|N| - 1$ edges, $G_2$ has exactly $|N| - 1$ edges, and thus $G_2 = G_1$.

If $b(1) < c$, then by Proposition 3, the best response network $G_2$ is not equal to $G_1$. By Lemma 1, $G_2$ cannot be a connected graph and thus has components $G_{2i} = (N_i, E_{2i})$. Denote the induced subgraphs of $G_1$ on vertex sets $N_i$ by $G_{1i} = (N_i, E_{1i})$. By Lemma 2, $G_{2i}$ is the best response network to $G_{1i}$. If $G_2$ is not the empty graph, $|N_i| \geq 2$ for some component $G_{2i}$. Since $G_1$ is a tree, $G_{1i}$ has $|N_i| - 1$ or fewer edges. Thus, by Lemma 1 and Proposition 3, $G_{2i}$ must have fewer than $|N_i| - 1$ edges. This contradicts the fact that $G_{2i}$ is a component, and thus $G_2$ must be the empty graph whenever $b(1) < c$.

Corollary 2: Let $G_1$ be a forest. Then $G_2 = G_1$ is the best response network to $G_1$ if $b(1) > c$. Otherwise, if $b(1) < c$, the empty network is the best response to $G_1$.

Proof: Denote the components of $G_1$ by $G_{1i} = (N_{1i}, E_{1i})$ for $i = 1, 2, \ldots, k$, where $k \geq 2$. Based on Lemma 1, we know that $G_2$ must be a disconnected graph. We want to show that the components of $G_2$ are the same as the components of $G_1$ if $b(1) > c$. Denote the components of $G_2$ by $G_{2i} = (N_{2i}, E_{2i})$ for $i = 1, 2, \ldots, m$. Suppose that there exists a $N_{2i}$ consisting of nodes from multiple $N_{1j}$. Denote the induced subgraph of $G_1$ on $N_{2i}$ by $G_{1i}$. According to Lemma 2, $G_{2i}$ is the best response network to
But we know that $G_{1}$ is a forest with more than one tree and consequently $G_{2}$ is not a connected graph, contradicting our assumption that $G_{2}$ is a component of $G_{2}$. Next we show that $N_{2i}$ cannot be a strict subset of $N_{1j}$. By way of contradiction, suppose that $N_{2i}$ is a strict subset of $N_{1j}$. Then there must exist an $N_{2i}$ that is also a strict subset of $N_{1j}$, such that there is an edge in $E_{1}$ between some node in $N_{2i}$ and some node in $N_{2j}$ (since graph $G_{1j}$ is connected). Since $b(1) > c$, as in the proof of the Proposition 1, adding this edge to $G_{2}$ increases the utility $u(G_{2}(G_{1}))$, contradicting the assumption that $G_{2}$ is the best response graph. Thus the node sets of the components of $G_{2}$ are the same as the node sets of the components of $G_{1}$. By Lemma 2 and Corollary 1, each component of $G_{2}$ is equal to the corresponding component of $G_{1}$, and thus $G_{2} = G_{1}$.

If $b(1) < c$, the same argument as in Corollary 1 shows that each $N_{2i}$ is a single vertex and as a result, the best response network to $G_{1}$ is the empty graph, which proves the claim. In both of the above cases, $G_{2}$ is equal to the union of the best response networks to each of the $G_{1i}$. ■

V. PAIRWISE STABILITY

In the previous section, we assumed that a centralized designer chooses the best response network to a given network. In this section, we study the satisfaction of the individual nodes in the network with the decision of the central designer. Specifically, let $G_{1} = (N, E_{1})$ be a given graph, and let $U$ denote the set of all possible utility functions $u(G_{2}(G_{1}))$ for graph $G_{2}$ based on $G_{1}$. For each $i \in N$, define the allocation rule $u_{i}(G_{2}, G_{1}, u) : G_{N} \times G_{N} \times U \rightarrow \mathbb{R}$ specifying the amount of utility that we allocate to player $i$ from the overall utility generated by the formed network $G_{2}$. For simplicity, we will use the notation $u_{i}(G_{2})$ when $G_{1}$ and $u$ are fixed.

For a given best response graph $G_{2}$ and individual utility functions $u_{i}$, it may be the case that a certain node can improve its own utility by removing one or more of its incident edges in $G_{2}$, or by adding additional edges from itself to other nodes. As in [8], we assume any node can remove any of its incident edges unilaterally, but that adding an edge to another node requires the consent of that node.

This motivates the following definition of pairwise stability of a given network [8]. In this definition, when $(i, j) \notin G$, $G + ij$ denotes the graph obtained by adding an edge between $i$ and $j$ in $G$. Similarly, $G - ij$ represents the graph obtained by deleting the edge $(i, j)$ when $(i, j) \notin G$.

Definition 1 ([8]): A graph $G = (N, E)$ is said to be pairwise stable if

- $\forall (i, j) \in E, u_{i}(g) \geq u_{i}(g - ij)$ and $u_{j}(g) \geq u_{j}(g - ij)$, and
- $\forall (i, j) \notin E$, if $u_{i}(g + ij) > u_{i}(g)$ then $u_{j}(g + ij) < u_{j}(g)$.

The graph is pairwise unstable if it is not pairwise stable. ■

In words, pairwise stability of a network corresponds to the situation where no node has any incentive to change any (one) of its connections in the network. This is a modification of the notion of a Nash equilibrium in network formation, capturing the concept of negotiation and agreement between the endpoints prior to forming the edge. Various versions of this notion have been studied in the network formation literature [3], [7], [16].

We now investigate the pairwise stability properties of the best response networks that were characterized in the previous section. Consider the allocation rule

$$u_{i}(G_{2}(G_{1})) = \frac{1}{2} \sum_{(i, v) \in E_{1}} b(d_{G_{2}(i, v))} - \frac{c}{2} \text{deg}_{i}(G_{2}),$$

where $\text{deg}_{i}(G)$ is the degree of node $i$ in graph $G$. Note that the total utility (2) satisfies $u(G_{2}(G_{1})) = \sum_{i \in N} u_{i}(G_{2}(G_{1}))$.

It is not hard to show that for any $i, j \in N$ where $G_{2} = (N, E_{2})$, if $(i, j) \notin E_{2}$, then it would not be beneficial for at least one of the nodes $i$ or $j$ to add the edge $(i, j)$ to the network $G_{2}$. By way of contradiction assume that $u_{i}(G_{2} + ij) \geq u_{i}(G_{2})$ and $u_{j}(G_{2} + ij) \geq u_{j}(G_{2})$ with one of the inequalities strict. Then, $u_{i}(G_{2} + ij) + u_{j}(G_{2} + ij) > u_{i}(G_{2}) + u_{j}(G_{2})$. However, this means that $u_{i}(G_{2} + ij) + u_{j}(G_{2})$ which contradicts the assumption that $G_{2}$ is the optimal network. This immediately implies that if the empty network is the best response to a graph, it is pairwise stable. For a general graph, to conclude that the best response network $G_{2}$ is pairwise stable, we also need to show that removing any of the edges from network $G_{2}$ is not beneficial for any of its endpoints. However, this is not true in general. As an example, consider Figure 1, where the edge $(1, 8)$ in $G_{2}$ is only useful for connecting nodes other than its endpoints. This network $G_{2}$ is not stable, since both nodes 1 and 8 could improve their utility by removing this edge.

The following result provides a condition under which the best response network obtained from solving the optimization problem (2) is pairwise stable with respect to the allocation rule (3).

Proposition 4: If the best response network with respect to network $G_{1}$ is $G_{2} = G_{1}$, then $G_{2}$ is pairwise stable.

Proof: As argued above, adding an edge is not beneficial to any node. Thus it suffices to show that removing any of the edges is unrewarding for both of its endpoints. Since $G_{2} = G_{1}$, edge $(i, j)$ is only useful for the connection between nodes $i$ and $j$. So if it is beneficial for one of the endpoints $i$ or $j$ to sever the link $(i, j)$, it is also beneficial for the other endpoint. Consequently, removing the edge $(i, j)$ increases the utility of $G_{2}$. This contradicts the fact that $G_{2} = G_{1}$ is the best response to $G_{1}$. ■

Remark 1: Note that the above result encompasses cases where $G_{1}$ is an arbitrary network and $b(1) = c > b(2)$ (by Proposition 2), and where $G_{1}$ is a tree (by Corollary 2). ■

VI. MULTI-LAYER NETWORKS WITH ARBITRARY UTILITIES

So far we have discussed the construction of one graph with respect to another graph based on the distance utility function, and how nodes of the network evaluate decisions made by the central designer. In this section, we consider the scenario where there is no central designer and nodes (players) themselves establish multiple different types of
relationships with other players over time; each type of relationship corresponds to a different layer (or edge set) on the set of nodes \( N \). The utility of the players is a function of their status in both networks \( G_1 \) and \( G_2 \). We will start by briefly reviewing the concept of an improving path [16] for single-layer network formation. Note that we are not assuming distance-based utility functions in this section and the analysis is applicable to any utility function.

### A. Single Layer Improving Path

To avoid confusion with the distance-based utility function, we will denote the utility function of a graph by \( v : G^N \rightarrow \mathbb{R} \) and allocation rule by \( Y_i(G,v) : G^N \times V \rightarrow \mathbb{R} \) where \( V \) is the set of all utility functions \( v \). For simplicity, we will omit the argument \( v \) in \( Y_i(G,v) \). As defined in [16], two networks \( G'_1 \) and \( G_1 \) are said to be adjacent if they only differ in one edge, i.e., \( \exists i,j \in N \) such that \( G'_1 = G_1 \pm ij \). For two adjacent graphs \( G'_1 \) and \( G_1 \), \( G'_1 \) defeats \( G_1 \) if either

1. \( G'_1 = G_1 + ij \), with \( Y_i(G'_1) \geq Y_i(G_1) \) and \( Y_j(G'_1) \geq Y_j(G_1) \) where at least one of the inequalities is strict, or
2. \( G'_1 = G_1 - ij \), with \( Y_i(G'_1) > Y_i(G_1) \) or \( Y_j(G'_1) > Y_j(G_1) \).

As in the previous section, a network is pairwise stable if it is not defeated by any of its adjacent networks. In other words, in a pairwise stable network, no node can increase its utility by severing one of its connections to other nodes in the network. Also, since we are assuming that consent of both endpoints is required to form a link, in a stable network, if it is the case that forming a link is beneficial for one of the endpoints, it is not for the other.

Now, imagine a graph called \( \Gamma \) with elements in the set \( G^N \) as its nodes. There is an edge from node \( G \) to node \( G' \) if they are adjacent and \( G' \) defeats \( G \). An improving path in \( \Gamma \) is a sequence of nodes \( G^{(1)}G^{(2)}\ldots G^{(k)} \) where \( G^{(i+1)} \) is adjacent to and defeats \( G^{(i)} \) for \( i < k \). According to the definition, a pairwise stable network is not defeated by any other network, and so no improving path emanates from a pairwise stable network in \( \Gamma \). The notion of an improving path corresponds to dynamic behavior of the players in the network formation game where at each step, one of the players removes one of his edges or two players decide to form an edge in such a way to maximize their utility given the other edges in the network [7]. Note that the players’ actions in this process are completely myopic because they do not consider future actions by other players. For example, while removing an edge might be beneficial in the current state, it could result in subsequent deletion or addition of other edges by other players which could reduce the utility of the original player.

The graph \( \Gamma \) is a directed graph. A strongly connected component (SCC) in the graph is a set of nodes such that each node in the set has a path to every other node in the set, and no node can be added to the set without breaking this property. A maximal SCC is an SCC where no node has an outgoing edge to any node not in the SCC. Every directed graph with a finite number of nodes has at least one maximal SCC [17]. Note that if a maximal SCC in \( \Gamma \) is a single node, the graph corresponding to that node is pairwise stable. On the other hand, if a maximal SCC contains multiple nodes, improving paths inside that set will cycle among the nodes in that set forever; such a set will be called a closed set of networks. This immediately leads to the following characterization of improving paths from [16].

**Lemma 3 ([16]):** For any value function \( v \) and allocation rule \( Y \), there exists at least one pairwise stable network or a closed set of networks.

### B. Improving Path for Two-Layer Network Formation Game

In this section, we generalize the idea of an improving path to strategic two-layer network formation.\(^1\) As mentioned earlier, here we assume that the nodes in the set \( N = \{1,2,\ldots,n\} \) are players of the game. A two layer network represented as \((G_1,G_2)\) is an outcome of the game where layer \( i \in \{1,2\} \) is a graph \( G_i = (N,E_i) \). Each player decides about her connections in different layers based on her utility which is a function of her connections in both layers of the network.

Similar to the single layer network formulation described earlier, there is a value function \( v(G_1,G_2) : G^N \times G^N \rightarrow \mathbb{R} \) that evaluates the pair \((G_1,G_2)\) and assigns to it a nonnegative real number. The utility of player \( i \) is represented by \( Y_i(G_1,G_2,v) \); we will omit the argument \( v \) and simply denote it as \( Y_i(G_1,G_2) \). Two pairs \((G_1,G_2)\) and \((G'_1,G'_2)\) are said to be adjacent if one could reach the new pair \((G'_1,G'_2)\) with at most one change in one of the graphs \( G_1 \) or \( G_2 \). In other words, either \( G'_1 = G_1 \pm ij \) or \( G'_2 = G_2 \pm ij \), but not both. We use the notation \((G'_1,G'_2) = (G_1,G_2) \pm (ij)^k \) for \( k = 1,2 \), depending on the layer from which we add or remove the edge \((i,j)\). We say that the pair \((G'_1,G'_2)\) defeats the pair \((G_1,G_2)\) if they are adjacent and one of the following conditions hold:

1. If \((G'_1,G'_2) = (G_1,G_2) + (ij)^k \) where \( k \in \{1,2\} \), with \( Y_i(G'_1,G'_2) \geq Y_i(G_1,G_2) \) and \( Y_j(G'_1,G'_2) \geq Y_j(G_1,G_2) \), where at least one of the inequalities is strict.
2. If \((G'_1,G'_2) = (G_1,G_2) - (ij)^k \) where \( k \in \{1,2\} \), with \( Y_i(G'_1,G'_2) > Y_i(G_1,G_2) \) or \( Y_j(G'_1,G'_2) > Y_j(G_1,G_2) \).

If a pair of networks \((G_1,G_2)\) is not defeated by any other adjacent pair of networks, we say it is intra-layer pairwise stable. Similar to the previous section, consider a directed graph \( \Gamma^2 \) with pairs \((G_1,G_2) \in G^N \times G^N \) as its nodes. There is an edge from node \((G_1,G_2)\) to node \((G'_1,G'_2)\), if they are adjacent and the former pair is defeated by the latter. An improving path in \( \Gamma^2 \) is a chain of pairs \((G_1,G_2),(G'_1,G'_2),\ldots(G_{l-1}',G_{l-1}'),G_{l+1}'\) where there is an edge from node \((G_1,G_2)\) to node \((G_{l+1}',G_{l+1}')\). Lemma 3 is still applicable here and thus for any \( v \) and \( Y \), there exists at least one intra-layer pairwise stable network or a closed set of two-layer networks in \( \Gamma^2 \).

\(^1\)It is worth noting that all of our analysis can be generalized to the \( m \)-layer network formation case; however for the sake of simplicity we only state the 2-layer case.
The following lemma relates intra-layer pairwise stable networks to stability properties of the individual layers when the utility function has a special form. We will use the following definition.

**Definition 2:** A function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is said to be increasing in its arguments (IA) if and only if for all \( a, b, c, d \in \mathbb{R} \),

1. \( f(a, b) > f(c, b) \Leftrightarrow a > c \).
2. \( f(a, b) > f(a, d) \Leftrightarrow b > d \).

**Lemma 4:** Suppose that the utility function of each node \( i \) has the form \( Y_i(G_1, G_2) = f(Y_i^1(G_1), Y_i^2(G_2)) \) where \( Y_i^k : G^k \times V \to \mathbb{R} \) is the utility function of node \( i \) in layer \( k \) and \( f \) is an IA function. Then the following statements are equivalent:

1. \( G_1^1 = (N, E_1^1) \) and \( G_2^2 = (N, E_2^2) \) are pairwise stable.
2. \( (G_1^1, G_2^2) \) is intra-layer pairwise stable.

**Proof:** Assume that statement 1 is true. By way of contradiction suppose that \( (G_1^1, G_2^2) \) is not intra-layer pairwise stable. Then one of the following cases must happen.

- \( \exists (i, j) \in E_1^1 \) with \( k \in \{1, 2\} \) such that \( Y_i((G_1^1, G_2^2) - (ij)^k) > Y_i(G_1^1, G_2^2) \). Since we are making a change in only one of the layers and the amount of utility that nodes receive from each of their layers is a function of the edge set in only that layer and \( f \) is IA, we can conclude that \( Y_i^1(G_1^1 - ij) > Y_i^1(G_1^1, G_2^2) \). However, this is impossible due to the assumption that \( G_1^1 \) is pairwise stable.

- \( \exists (i, j) \notin E_1^1 \) such that \( Y_i((G_1^1, G_2^2) + (ij)^k) \geq Y_i((G_1^1, G_2^2)) \) and \( Y_i((G_1^1, G_2^2) + (ij)^k) \geq Y_i((G_1^1, G_2^2)) \) with one of the inequalities strict. Again since the change is in only one the layers and using the IA property, we can conclude that

\[
Y_i^1(G_1^1 + ij) \geq Y_i^1(G_1^1, G_2^2) \\
Y_i^k(G_2^2 + ij) \geq Y_i^k(G_2^2),
\]

with one of the inequalities strict. However, this contradicts the assumption that \( G_1^1 \) is pairwise stable.

Next we show that if statement 2 is true, statement 1 is also true. Again by way of contradiction suppose that \( (G_1^1, G_2^2) \) is intra-layer pairwise stable but \( G_1^1 \) is not pairwise stable.

Then one the following cases must happen.

- \( \exists (i, j) \in E_1^1 \) such that \( Y_i^1(G_1^1 - ij) > Y_i^1(G_1^1) \). Then based on the IA property of \( f \), we must have

\[
f(Y_i^1(G_1^1 - ij), Y_i^2(G_2^2)) > f(Y_i^1(G_1^1), Y_i^2(G_2^2)) \Rightarrow Y_i((G_1^1, G_2^2) - (ij)^k) > Y_i(G_1^1, G_2^2),
\]

which contradicts the intra-layer pairwise stability assumption of \( (G_1^1, G_2^2) \).

- \( \exists (i, j) \notin E_1^1 \) such that

\[
Y_i^1(G_1^1 + ij) \geq Y_i^1(G_1^1) \\
Y_i^k(G_2^2 + ij) \geq Y_i^k(G_2^2),
\]

with one of the inequalities strict. Then using the IA property, we can conclude that \( Y_i((G_1^1, G_2^2) + (ij)^1) \geq Y_i(G_1^1, G_2^2) \) and \( Y_j((G_1^1, G_2^2) + (ij)^1) \geq Y_j(G_1^1, G_2^2) \) with one of the inequalities strict, which again contradicts the intra-layer pairwise stability assumption of \( (G_1^1, G_2^2) \).

The same holds for \( G_2^2 \) and thus concludes the proof. ■

**VII. SUMMARY**

We studied the problem of strategic multi-layer network formation. We started by generalizing distance-based network formation to the multi-layer setting, where the edge set of a network is chosen with respect to another edge set on the same set of nodes (which can be represented as a graph). We characterized efficient networks in this setting with respect to some specific graphs and regions of benefit function \( b(\cdot) \) and edge cost \( c \). We showed that the best response networks will be pairwise stable whenever they are equal to the original network. Finally, we studied multi-layer network formation with general utilities, where each node is a player in the game. We showed that under certain conditions on the utility function, the stability of the multi-layer network can be related to the stability of each of the layers in the network.

**REFERENCES**