A Graph Theoretic Approach to the Robustness of k-Nearest Neighbor Vehicle Platoons

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***Abstract*—We consider a graph-theoretic approach to the performance and robustness of a platoon of vehicles, where each vehicle communicates with its k-nearest neighbors. In particular, we quantify the platoon's stability margin, robustness to disturbances (in terms of system $\mathcal{H}_\infty$ norm), and maximum delay tolerance via graph-theoretic notions such as nodal degrees and (grounded) Laplacian matrix eigenvalues. The results show that there is a trade-off between robustness to time delay and robustness to disturbances. Both first-order dynamics (reference velocity tracking) and second-order dynamics (controlling inter-vehicular distance) are analyzed in this direction. Theoretical contributions are confirmed via simulation results.**

I. Introduction

Recent advances in connecting system-theoretic notions (such as stability and robustness) to network properties have found various applications in networked systems [1]. One important application is in connected and cooperative vehicles [2], which will greatly impact the future of urban transportation [3]. Among the different applications of connected vehicles, cooperative cruise control has attracted much attention. This method is concerned with controlling vehicles' velocities to minimize fuel consumption and maintain prescribed inter-vehicular distances. One of the main challenges is to make these control policies resilient to external disturbances and to time delays in inter-vehicle communications. The aim of this paper is to present conditions for the robustness of a generalized form of vehicle platooning (called k-nearest neighbor platoon) to external disturbances and time delay.

Much effort has been made in analyzing the robustness of vehicle platoons to communication disturbances and among them is the well-known notion of string stability [4]. String stability occurs if the transfer function from disturbance in the first vehicle in a platoon to state error in the last vehicle has a bounded frequency magnitude peak independent of the platoon size [5]. The notion of robustness was revisited later in terms of network coherence in the control theory literature [6], where it was shown that for 1-D network topologies, it is impractical to have large coherent platoon using only local feedback. Alternatively, optimal controllers are designed in [7] to improve the coherence of a vehicle formation. The effect of time delay in vehicle communication on performance and stability in vehicle platoons was studied in [8], [9].

This paper is concerned with robustness analysis of vehicle platoons to delay and communication disturbances under two policies. First we analyze the velocity tracking scenario, which is applied to cases where the vehicle fuel consumption is to be minimized [10], [11]. Second, we analyze the network formation problem, where inter-vehicular distances are regulated to avoid collisions. In contrast with other works on robustness of vehicle platoons [12]–[15], here we present graph-theoretic robustness conditions for both of the above communication policies. We then analyze the effect of the number and locations of reference vehicles (leaders) on the robustness of the vehicle network. More specifically, the contributions of this paper are the following:

- Graph-theoretic bounds are provided for the system $\mathcal{H}_\infty$ norm of the *velocity tracking scenario*, in terms of network structure and the location of reference vehicles. Moreover, we provide an exact threshold on the maximum communication time-delay $\tau_{\max}$ which can be tolerated before the dynamics become unstable.
- Graph-theoretic bounds are presented for the system $\mathcal{H}_\infty$ norm of the *network formation problem*, and provide an upper bound on the maximum tolerable communication delay $\tau_{\max}$.

Compared to recent researches on the robustness of vehicle platoons such as [2], [14], [15], the current paper addresses the robustness to both external disturbances and time delay from network-theoretic standpoint. Moreover, the platooning models introduced in the literature were limited to 1-nearest neighbor network with a single reference vehicle. By introducing k-nearest neighbor platoons and considering multiple reference vehicles, we generalize the structure of the network of vehicles which enables us to analyze the effect of network topology and the location of reference vehicles (network control inputs) on the performance of the platooning.

The paper is organized as follows. Section II establishes some notation, before we introduce the network dynamics for both *velocity tracking* and *network formation* in Section III. Section IV briefly presents some graph-theoretic bounds on the eigenvalues of the grounded Laplacian matrix. Sections V and VI contain main results on the robustness of k-nearest neighbor platoons for velocity tracking and network formation scenarios. In Section VII some simulation results illustrate the proposed theory. Section VIII concludes the paper.

II. Notations and Definitions

A undirected graph (network) is denoted by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ is a set of nodes (or vertices) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. Neighbors of node $v_i \in \mathcal{V}$ are given by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The adjacency matrix of the graph is given by a symmetric and binary $n \times n$ matrix A, where element $A_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and zero otherwise. The degree of node v_i

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is denoted by $d_i \triangleq \sum_{j=1}^{n} A_{ij}$. For a given set of nodes $X \subset \mathcal{V}$, the *edge-boundary* (or just boundary) of the set is given by $\partial X \triangleq \{(v_i, v_j) \in \mathcal{E} \mid v_i \in X, v_j \in \mathcal{V} \setminus X\}$. The Laplacian matrix of the graph is given by $\mathcal{L} \triangleq D - A$, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. The eigenvalues of the Laplacian are real and nonnegative, and are denoted by $0 = \lambda_1(\mathcal{L}) \leq \lambda_2(\mathcal{L}) \leq \ldots \leq \lambda_n(\mathcal{L})$. For a given subset $\mathcal{S} \subset \mathcal{V}$ of nodes (which we term *grounded nodes*), the *grounded Laplacian* induced by $\mathcal{S}$ is denoted by $\mathcal{L}_g$, and is obtained by removing the rows and columns of $\mathcal{L}$ corresponding to the nodes in $\mathcal{S}$. In this paper, *grounded nodes* represent *reference vehicles*. When the network is connected and there exists at least one grounded node, the grounded Laplacian matrix $\mathcal{L}_g$ is a positive definite matrix [16]. For a given set $\mathcal{I}$, the number of members (cardinality) of the set is denoted by $|\mathcal{I}|$. For any $a \in \mathbb{R}$, $\lceil a \rceil$ is the smallest integer greater than or equal to a and $\lfloor a \rfloor$ is the largest integer less than or equal to a.

III. Problem Statement

Consider a connected network of n vehicles $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. Each vehicle $v_i \in \mathcal{V}$ is either a follower $v_i \in \mathcal{F}$ or a reference vehicle $v_i \in \mathcal{R}$. The position and longitudinal velocity of each vehicle v_i is denoted by scalars p_i and u_i, respectively. Vehicle speed can be measured by GPS or estimated by model-based [17] or combined [18] approaches. In this paper $\mathcal{P}(n, k)$ denotes a platoon of n vehicles where each vehicle can communicate with its k nearest neighbors from its back and k nearest neighbors from its front, for some $k \in \mathbb{N}$. This is due to the limited sensing and communication range for each vehicle and the distance between the consecutive vehicles. An example of $\mathcal{P}(n, k)$ is shown in Fig. 1. In Figure 1, each vehicle is connected to its two nearest neighbors. For instance, vehicle shown in grey color is connected to its two (nearest) neighbors from its back and two nearest neighbors from its front. Two

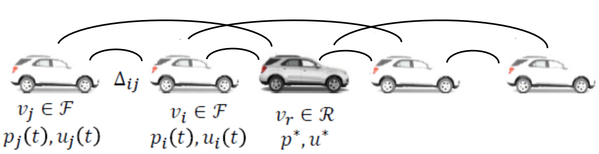

Fig. 1: A 2-nearest neighbor platoon of 5 vehicles, $\mathcal{P}(5, 2)$. A reference vehicle is located in the middle.

control objectives are addressed in this paper: (i) control the velocity of the vehicles (velocity tracking) or (ii) regulate the distance between neighboring vehicles (network formation). Each vehicle v_i is governed by the second order dynamics $\ddot{p}_i(t) = q_i(t)$, or in vector notation

$$\ddot{\mathbf{p}}(t) = \mathbf{q}(t), \tag{1}$$

where $\mathbf{p}(t) = [p_1(t), p_2(t), ..., p_n(t)]^T$ is the vector of positions and $\mathbf{q}(t)$ is the vector of control laws for either of the two above control objectives discussed in the following subsections.

A. Velocity Tracking

In this case, each follower vehicle attempts to track a reference velocity u^*. This desired reference velocity is calculated by reference vehicles in order to minimize fuel consumption. This yields the following control laws for each follower and reference vehicle, [19]

$$q_i(t) = \begin{cases} \sum_{j \in \mathcal{N}_i} k_u(u_j(t) - u_i(t)) & \forall v_i \in \mathcal{F}, \\ 0 & \forall v_i \in \mathcal{R}, \end{cases} \tag{2}$$

where $k_u > 0$ is the control gain. The state (velocity) of the reference vehicles (which should be tracked by the followers) is assumed to be constant and is not affected by other vehicles.

Remark 1: One can define control law $q_i(t) = \kappa(u^* - u_i(t))$ for all $v_i \in \mathcal{R}$ where u^* is the reference velocity. For sufficiently large κ it can be shown (by singular perturbation analysis) that the assumption $q_i(t) = 0$ in (2) is valid. □

Aggregating the velocities of all followers into a vector $\mathbf{u}_{\mathcal{F}}(t) \in \mathbb{R}^{|\mathcal{F}|}$, and the velocities of all reference vehicles into a vector $\mathbf{u}_{\mathcal{R}}(t) \in \mathbb{R}^{|\mathcal{R}|}$, one can write the following dynamics from (1) and (2):

$$\begin{bmatrix} \dot{\mathbf{u}}_{\mathcal{F}}(t) \\ \dot{\mathbf{u}}_{\mathcal{R}}(t) \end{bmatrix} = -k_u \underbrace{\begin{bmatrix} \mathcal{L}_g & \mathcal{L}_{12} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} \mathbf{u}_{\mathcal{F}}(t) \\ \mathbf{u}_{\mathcal{R}}(t) \end{bmatrix}, \tag{3}$$

where $\mathcal{L}_g \in \mathbb{R}^{|\mathcal{F}| \times |\mathcal{F}|}$ is the grounded Laplacian matrix, formed by removing the rows and columns of $\mathcal{L}$ corresponding to the reference vehicles. The control law for the follower vehicles in vector form becomes

$$\dot{\mathbf{u}}_{\mathcal{F}}(t) = -k_u \mathcal{L}_g \mathbf{u}_{\mathcal{F}}(t) - k_u \mathcal{L}_{12} \mathbf{u}_{\mathcal{R}}(0). \tag{4}$$

Introducing the deviation variable $\tilde{\mathbf{u}}_{\mathcal{F}}(t) = \mathbf{u}_{\mathcal{F}}(t) - \mathbf{u}_{\mathcal{F}}^{ss}$, where $\mathbf{u}_{\mathcal{F}}^{ss} = \lim_{t \to \infty} \mathbf{u}_{\mathcal{F}}(t)$, the error dynamics of (4) becomes

$$\dot{\tilde{\mathbf{u}}}_{\mathcal{F}}(t) = -k_u \mathcal{L}_g \tilde{\mathbf{u}}_{\mathcal{F}}(t). \tag{5}$$

The following proposition shows that each component of $\mathbf{u}_{\mathcal{F}}^{ss}$ is a weighted average of the reference velocities.

Proposition 1: Consider a platoon $\mathcal{P}(n, k)$ of vehicles with reference vehicle set $\mathcal{R}$ and follower set $\mathcal{F}$. Let $\mathcal{U} = \{u_1^*, u_2^*, ..., u_{|\mathcal{R}|}^*\}$ be the set of the reference velocities. Then each component of $\mathbf{u}_{\mathcal{F}}^{ss}$ is a convex combination of the elements of $\mathcal{U}$.

Proof: The unique steady-state solution of (4) is $\mathbf{u}_{\mathcal{F}}^{ss} = -\mathcal{L}_g^{-1} \mathcal{L}_{12} \mathbf{u}_{\mathcal{R}}(0)$. We know that $\mathcal{L}\mathbf{1} = \mathbf{0}$ which yields $\mathcal{L}_g \mathbf{1}_{|\mathcal{F}| \times 1} + \mathcal{L}_{12} \mathbf{1}_{|\mathcal{R}| \times 1} = \mathbf{0}_{|\mathcal{F}| \times 1}$ and that results in $-\mathcal{L}_g^{-1} \mathcal{L}_{12} \mathbf{1}_{|\mathcal{R}| \times 1} = \mathbf{1}_{|\mathcal{F}| \times 1}$. Hence, $-\mathcal{L}_g^{-1} \mathcal{L}_{12}$ is a row stochastic matrix, and each component of $\mathbf{u}_{\mathcal{F}}^{ss}$ is a convex combination of the velocities of reference vehicles. ■

Based on Proposition 1, for the case where the velocities of all reference vehicles are the same, i.e. $\mathbf{u}_{\mathcal{R}}(t) = u^* \mathbf{1}_{|\mathcal{R}| \times 1}$, the velocity of all followers reach to u^*.[1]

[1] In this paper we assume that all reference vehicles have the same velocity u^* and followers track this unique reference velocity.

B. Network Formation

In this case, the objective for each follower vehicle is to maintain specific distances from its neighbor vehicles. The desired vehicle formation will be formed by a specific constant distance Δ_{ij} between vehicles v_i and v_j, which should satisfy $\Delta_{ij} = \Delta_{ik} + \Delta_{kj}$ for every triple $\{v_i, v_j, v_k\} \subset \mathcal{V}$. Considering the fact that each vehicle v_i has access to its own position, the positions of its neighbors, and the desired inter-vehicular distances Δ_{ij}, the control law for vehicle v_i is [20]

$$q_i(t) = \sum_{j \in \mathcal{N}_i} k_p \left(p_j(t) - p_i(t) + \Delta_{ij}\right) + k_u \left(u_j(t) - u_i(t)\right), \tag{6}$$

where $k_p, k_u > 0$ are control gains. We define the tracking error $\tilde{p}_i(t) = p_i(t) - p_i^*(t)$, where $p_i^*(t)$ is the desired trajectory of vehicle v_i which should satisfy $\Delta_{ij} = p_i^*(t) - p_j^*(t)$ for all $v_i, v_j \in \mathcal{V}$. By rewriting (6) we have

$$\ddot{\tilde{p}}_i(t) = \sum_{j \in \mathcal{N}_i} k_p \left(\tilde{p}_j(t) - \tilde{p}_i(t)\right) + k_u \left(u_j(t) - u_i(t)\right), \tag{7}$$

which comes from the fact that the rigid formation requires $u_j^*(t) = u_i^*(t)$ which results in $\tilde{u}_j(t) - \tilde{u}_i(t) = u_j(t) - u_i(t)$, where $\tilde{u}_i(t)$ and $\tilde{u}_j(t)$ are i-th and j-th elements of $\tilde{\mathbf{u}}_{\mathcal{F}}(t)$.

Remark 2: A more realistic version of vehicle spacing policy is *velocity adaptable spacing*, which determines the desired vehicle spacing Δ_{ij} based on velocities v_i and v_j. However, in this paper we assume that vehicle velocities do not change much during the time when the network dynamics is operating. This is a reasonable assumption, since inter-vehicular communications are fast enough (data is transmitted with the rate of 10 Mbps) which results in fast network formation of the platoon. Hence Δ_{ij} is set to be constant, based on (constant) velocities v_i and v_j. □

The error dynamics (7) in state-space form are

$$\dot{\tilde{\mathbf{x}}}_{\mathcal{F}}(t) = \mathcal{B}\tilde{\mathbf{x}}_{\mathcal{F}}(t), \tag{8}$$

where $\tilde{\mathbf{x}}_{\mathcal{F}} = [\tilde{\mathbf{P}}_{\mathcal{F}} \; \dot{\tilde{\mathbf{P}}}_{\mathcal{F}}]^T = [\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_{|\mathcal{F}|}, \dot{\tilde{p}}_1, \dot{\tilde{p}}_2, ..., \dot{\tilde{p}}_{|\mathcal{F}|}]^T$ and $\mathcal{B} = I_{|\mathcal{F}| \times |\mathcal{F}|} \otimes \mathcal{B}_1 + \mathcal{L}_g \otimes \mathcal{B}_2$, in which $\otimes$ is Kronecker product and

$$\mathcal{B}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 0 & 0 \\ -k_p & -k_u \end{bmatrix}. \tag{9}$$

Remark 3: Going forward we assume $k_p = k_u = 1$ and we focus on the effect of the network structure (not control gains) on the robustness of vehicle platoon. The results can be easily extended for all $k_p, k_u > 0$. □

The following theorem specifies the spectrum of matrix $\mathcal{B}$ in (8) in terms of the spectrum of $\mathcal{L}_g$.

Theorem 1 ([20]): The spectrum of $\mathcal{B}$, $\sigma(\mathcal{B})$, is

$$\sigma(\mathcal{B}) = \cup_{\lambda_i \in \sigma(\mathcal{L}_g)} \sigma \underbrace{\begin{bmatrix} 0 & 1 \\ -\lambda_i & -\lambda_i \end{bmatrix}}_{\mathcal{D}}. \tag{10}$$

Thus by forming the characteristic polynomial of matrix $\mathcal{D}$ in (10) we have

$$\lambda_i(\mathcal{B}) = \begin{cases} -\frac{\lambda_i(\mathcal{L}_g)}{2}\left(1 + \left(1 - \frac{4}{\lambda_i(\mathcal{L}_g)}\right)^{\frac{1}{2}}\right), & 1 \le i \le |\mathcal{F}|, \\ -\frac{\lambda_{i-|\mathcal{F}|}(\mathcal{L}_g)}{2}\left(1 - \left(1 - \frac{4}{\lambda_{i-|\mathcal{F}|}(\mathcal{L}_g)}\right)^{\frac{1}{2}}\right), & i > |\mathcal{F}|, \end{cases} \tag{11}$$

where $i = 1, 2, ..., 2|\mathcal{F}|$. Based on the fact that $\lambda_{i-|\mathcal{F}|}(\mathcal{L}_g)$ for $|\mathcal{F}| \le i \le 2|\mathcal{F}|$ is the same as $\lambda_i(\mathcal{L}_g)$ for $1 \le i \le |\mathcal{F}|$, each eigenvalue of $\mathcal{L}_g$ in (11) forms two eigenvalues of $\mathcal{B}$ and since $\mathcal{L}_g$ is a positive definite matrix, the real parts of all of the eigenvalues of $\mathcal{B}$ are negative. □

C. Robustness Notions for Vehicle Platoons

From now on we refer to the error dynamics (5) as *velocity tracking dynamics* and to (8) as *network formation dynamics*. These control policies are prone to imprecisions due to the inter-vehicle communication disturbances or other uncertainties such as road friction condition. Hence, both velocity tracking dynamics and network formation dynamics can be written in the following form

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{J}\mathbf{w}(t), \tag{12}$$

where $\mathbf{w}(t)$ is a vector which represents bounded disturbances. Here $\mathcal{A} = -\mathcal{L}_g$, $\mathcal{J} = I_{|\mathcal{F}| \times |\mathcal{F}|}$ for the velocity tracking dynamics and $\mathcal{A} = \mathcal{B}$, $\mathcal{J} = [\mathbf{0}_{|\mathcal{F}| \times |\mathcal{F}|} \quad I_{|\mathcal{F}| \times |\mathcal{F}|}]^T$ for the network formation dynamics. As output signals of interest, we consider *velocity deviation* for the velocity tracking dynamics and *position deviation* for the network formation dynamics, as shown in Fig. 2. Based on the input-output representation

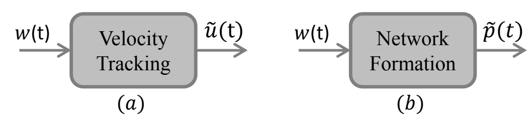

Fig. 2: Input/outputs of error dynamics (a) (5) and (b) (8).

of both dynamics, the robustness of the vehicle platoon to disturbances is quantified using the $\mathcal{H}_\infty$ norm of the transfer function from the disturbances to the output signals.

Remark 4: The notion of system $\mathcal{H}_\infty$ norm discussed in this paper is to address the robustness of each agent's state error (position or velocity) to external disturbances. Thus, it is different from the notion of $\mathcal{L}_2$ string stability [4], which addresses the effect of the disturbances on the first vehicle to the state error of the last vehicle in a platoon. □

In addition to disturbances, the inter-vehicle communication is prone to time delay which may inhibit tracking or even cause instability. To model this, the updating policies (5) and (8) can be modified as

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t - \tau), \tag{13}$$

where $\tau \in [0, \tau_{\max}]$ is a constant, bounded time delay.

In the following section, a brief overview of the spectrum of the grounded Laplacian matrix is presented.

IV. Smallest and Largest Eigenvalues of $\mathcal{L}_g$

Spectrum of $\mathcal{L}_g$ has a pivotal role in the performance and robustness of the velocity tracking and network formation dynamics (5) and (8). The following theorem provides bounds on the smallest eigenvalue $\lambda_1(\mathcal{L}_g)$ and the largest eigenvalue $\lambda_{|\mathcal{F}|}(\mathcal{L}_g)$ in terms of network properties.

Theorem 2 ([16], [21]): Consider a connected network $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with a set of reference vehicles $\mathcal{R} \subset \mathcal{V}$. Let $\mathcal{L}_g$ be the grounded Laplacian matrix for $\mathcal{G}$. Let $\beta_i = |\mathcal{N}_i \cap \mathcal{R}|$ be the number of reference vehicles in follower v_i's neighborhood. Then the smallest eigenvalue $\lambda_1(\mathcal{L}_g)$ of $\mathcal{L}_g$ satisfies

$$\min_{i\in\mathcal{F}}\{\beta_i\} \leq \lambda_1(\mathcal{L}_g) \leq \frac{|\partial\mathcal{R}|}{|\mathcal{F}|} \leq \max_{i\in\mathcal{F}}\{\beta_i\} \leq |\mathcal{R}|, \qquad (14)$$

and the largest eigenvalue $\lambda_{|\mathcal{F}|}(\mathcal{L}_g)$ of $\mathcal{L}_g$ satisfies

$$d_{\max}^{\mathcal{F}} \leq \lambda_{|\mathcal{F}|}(\mathcal{L}_g) \leq 2d_{\max}^{\mathcal{F}}, \qquad (15)$$

where $d_{\max}^{\mathcal{F}} = \max_{v_i\in\mathcal{F}} d_i$, is the maximum degree over the follower vehicles. □

V. Robustness of Velocity Tracking Dynamics

In Section IV, some useful spectral properties of the grounded Laplacian matrix $\mathcal{L}_g$ were introduced. In this section, we use those results to give graph-theoretic conditions for the stability margin of the velocity tracking dynamics (5) and its robustness to disturbances and time delay.

A. Stability Margin and Robustness to Disturbances

The stability margin of (5) is determined by $\lambda_1(\mathcal{L}_g)$. Hence, the graph-theoretic bounds provided in (14) can be considered as bounds on the stability margin accordingly. Now suppose that, as described above (12), the velocity tracking dynamics (5) are subject to an external disturbance $\mathbf{w}(t)$, yielding

$$\dot{\tilde{\mathbf{u}}}_{\mathcal{F}}(t) = -\mathcal{L}_g\tilde{\mathbf{u}}_{\mathcal{F}}(t) + \mathbf{w}(t). \qquad (16)$$

The transfer function of (16) is $G(s) = (sI + \mathcal{L}_g)^{-1}$. Here we measure system robustness using the $\mathcal{H}_\infty$ norm of (16), defined as $||G||_\infty \triangleq \sup_{\omega\in\mathbb{R}} \lambda_{max}^{\frac{1}{2}}(G^*(j\omega)G(j\omega))$ [22].

Proposition 2 ([23]): The system $\mathcal{H}_\infty$ norm of (16) is $||G||_\infty = \frac{1}{\lambda_1(L_g)}$.

Based on Proposition 2 and Theorem 2, the following bounds for $\mathcal{H}_\infty$ norm of (16) can be written:

$$\frac{1}{\max_{i\in\mathcal{F}}\{\beta_i\}} \leq \frac{|\mathcal{F}|}{|\partial\mathcal{R}|} \leq ||G||_\infty \leq \frac{1}{\min_{i\in\mathcal{F}}\{\beta_i\}}. \qquad (17)$$

For the case where $\min_{i\in\mathcal{F}}\{\beta_i\} = 0$, the upper bound in (17) is infinity. According to (17), we have the following corollary.

Corollary 1: Consider a vehicle platoon $\mathcal{P}(n,k)$ with reference vehicle set $\mathcal{R}$ and follower set $\mathcal{F}$. For any $\gamma > 0$, necessary and sufficient conditions for $\mathcal{P}(n,k)$ to have $||G||_\infty < \gamma$ in velocity tracking dynamics (5) are $\max_{i\in\mathcal{F}}\{\beta_i\} > \lfloor\frac{1}{\gamma}\rfloor$ and $\min_{i\in\mathcal{F}}\{\beta_i\} > \lceil\frac{1}{\gamma}\rceil$, respectively.

Considering Corollary 1, we can propose graph-theoretic necessary and sufficient conditions for a vehicle platoon $\mathcal{P}(n,k)$ to have a non-expansive system $\mathcal{H}_\infty$ norm (i.e. $||G||_\infty \leq 1$). This is an important feature of the network control system which ensures that the system is able to attenuate external disturbances. To address this, we present the following theorem which introduces necessary and sufficient conditions for the number of reference vehicles in $\mathcal{P}(n,k)$ to have a non-expansive system $\mathcal{H}_\infty$ norm. Before that, a specific arrangement of the reference vehicles in the platoon is introduced.

Definition 1: An arrangement of reference vehicles is called *minimally dense* (MD) if $\mathcal{P}(n,k)$ is partitioned into line segments with length $2k+1$ starting from one end such that in the *middle* of each partition one reference vehicle is located (which will be connected to all of the followers in that partition). □

Based on MD arrangement there exist $\lceil\frac{n}{2k+1}\rceil$ reference vehicles in $\mathcal{P}(n,k)$. As an example, the grey color vehicle in Fig. 1 is set in $\mathcal{P}(5,2)$ based on MD arrangement policy, considering $\lceil\frac{n}{2k+1}\rceil = 1$ for $\mathcal{P}(5,2)$. The following theorem introduces conditions for $\mathcal{P}(n,k)$ to have non-expansive $\mathcal{H}_\infty$ norm.

Theorem 3: Consider a k-nearest neighbor platoon $\mathcal{P}(n,k)$ with dynamics (16). If there exist at least $|\mathcal{R}| = \lceil\frac{n}{2k+1}\rceil$ reference vehicles, then there exists an arrangement of the reference vehicles satisfying $||G||_\infty \leq 1$. Moreover if the number of reference vehicles is less than $\lceil\frac{n}{2k+1}\rceil$, then there is no arrangement of reference vehicles satisfying $||G||_\infty \leq 1$. □

Proof: First, the sufficient condition is explored. Based on Corollary 1, a sufficient condition for $||G||_\infty \leq 1$ is to have $\min_{i\in\mathcal{F}}\{\beta_i\} \geq 1$. By doing an MD arrangement of $\lceil\frac{n}{2k+1}\rceil$ reference vehicles in $\mathcal{P}(n,k)$ we will have $\min_{i\in\mathcal{F}}\{\beta_i\} \geq 1$.

Next we have to show that with less than this number of reference vehicles, it is impossible to obtain $||G||_\infty \leq 1$. From a lower bound in (17), a necessary condition for $||G||_\infty \leq 1$ is to have $\frac{|\mathcal{F}|}{|\partial\mathcal{R}|} \leq 1$. Based on the fact that $|\partial\mathcal{R}| \leq 2k|\mathcal{R}|$ we have $\frac{|\mathcal{F}|}{2k|\mathcal{R}|} \leq \frac{|\mathcal{F}|}{|\partial\mathcal{R}|}$. Thus $\frac{|\mathcal{F}|}{2k|\mathcal{R}|} \leq 1$ is a necessary condition for $||G||_\infty \leq 1$, which yields $|\mathcal{R}| \geq \frac{n}{2k+1}$. ■

Based on Theorem 3, the MD arrangement of reference vehicles in $\mathcal{P}(n,k)$ provides the minimum possible number of reference vehicles to yield a non-expansive $\mathcal{H}_\infty$ norm.

B. Robustness to Time Delay

Here we discuss the stability of dynamics (5) when the vehicles control policies are subject to a constant time delay $\tau \in [0, \tau_{\max}]$. The stability of the linear system (5) to time delay is discussed in the following theorem, which is based on a general result in [24].

Theorem 4: The velocity tracking dynamics (5) is asymptotically stable in the presence of constant time delay $\tau \in [0, \tau_{\max}]$ if and only if

$$\tau_{\max} < \min_i\left\{\frac{\pi}{2\lambda_i(\mathcal{L}_g)}\right\} = \frac{\pi}{2\lambda_{|\mathcal{F}|}(\mathcal{L}_g)}. \qquad (18)$$

□

Based on Theorems 2 and 4, the following proposition introduces graph-theoretic necessary and sufficient conditions for the stability of $\mathcal{P}(n,k)$ under time delay.

Proposition 3: A vehicle platoon $\mathcal{P}(n,k)$ under velocity tracking dynamics (5) in the presence of constant time delay

$\tau \in [0, \tau_{\max}]$ is asymptotically stable if $\tau_{\max} \leq \frac{\pi}{8k}$ and it is unstable if $\tau_{\max} > \frac{\pi}{2k}$.

Proof: According to Theorems 2 and 4, necessary and sufficient conditions for asymptotic stability of (5) in the presence of time delay are $\tau_{\max} < \frac{\pi}{2d^{\mathcal{F}}_{\max}}$ and $\tau_{\max} < \frac{\pi}{4d^{\mathcal{F}}_{\max}}$, respectively, and according to the fact that $k \leq d^{\mathcal{F}}_{max} \leq 2k$ the results are obtained. ■

Remark 5: Corollary 1 and Proposition 3 show that there is a trade-off between robustness to disturbances and time delay when the connectivity index k increases. In particular, by increasing k (for fixed number of reference vehicles) $\min_{i\in\mathcal{F}}\{\beta_i\}$ increases while the maximum delay $\tau_{\max}$ decreases. □

VI. Robustness of Network Formation Dynamics

This section analyzes the performance and robustness of the network formation dynamics (8). For the sake of compatibility with the previous section, here we assume that reference vehicles are arranged based on the MD arrangement, discussed in Section V.

A. Stability Margin and Robustness to Uncertainty

For a k-nearest neighbor platoon $\mathcal{P}(n, k)$, the following proposition characterizes the stability margin of the platoon under the network formation dynamics (8) for an MD arrangement of reference vehicles. This stability margin is quantified using the least-stable eigenvalue of the system matrix $\mathcal{B}$.

Proposition 4: For the stability margin of the network formation dynamics (8) for the platoon $\mathcal{P}(n, k)$ with a MD arrangement of reference vehicles, we have

$$|\text{Re}(\lambda_1(\mathcal{B}))| = \frac{\lambda_1(\mathcal{L}_g)}{2}. \tag{19}$$

Proof: According to Theorem 2, for MD arrangement of reference vehicles, we have $\min_{i\in\mathcal{F}}\{\beta_i\} = 1$ and $\max_{i\in\mathcal{F}}\{\beta_i\} \leq 2$, where $\min_{i\in\mathcal{F}}\{\beta_i\}$ and $\max_{i\in\mathcal{F}}\{\beta_i\}$ are defined in (14). Hence, we have $1 \leq \lambda_1(\mathcal{L}_g) \leq 2$. Based on Theorem 1 and (11), for the eigenvalue of $\mathcal{B}$ with the smallest magnitude of the real part we have $1 - \frac{4}{\lambda_1(\mathcal{L}_g)} \leq 0$ which results in $|\text{Re}(\lambda_1(\mathcal{B}))| = \frac{\lambda_1(\mathcal{L}_g)}{2}$. ■

The following theorem gives an upper bound for the $\mathcal{H}_\infty$ norm of the network formation dynamics under the MD arrangement of reference vehicles.

Theorem 5: Consider $\mathcal{P}(n, k)$ with a MD arrangement of reference vehicles. The $\mathcal{H}_\infty$ norm from disturbances to the position error of (8) satisfies $||G||_\infty \leq \frac{2}{\sqrt{3}}$. □

Proof: Taking Laplace transform of (7) for zero initial conditions gives the following transfer function from disturbances to the position output

$$G(s) = \left(s^2 I + (s + 1)\mathcal{L}_g\right)^{-1}. \tag{20}$$

The transfer function (20) can be put in diagonal form as

$$\begin{aligned}\left(s^2 I + (s + 1)\mathcal{L}_g\right)^{-1} &= M\left(s^2 I + (s + 1)\Lambda\right)^{-1} M^T \\ &= M \,\text{diag}(G_i(s)) M^T,\end{aligned} \tag{21}$$

where $M = [v_1, v_2, ..., v_{|\mathcal{F}|}]$ is a matrix formed by eigenvectors of $\mathcal{L}_g$ and $\text{diag}(G_i(s))$ is a diagonal matrix with diagonal elements $G_i(s) = \frac{1}{s^2+\lambda_i(\mathcal{L}_g)s+\lambda_i(\mathcal{L}_g)}$ with the following maximum amplitudes

$$\mathcal{C}_i = \sup_\omega |G_i(j\omega)| = \begin{cases} \frac{2}{\lambda_i(\mathcal{L}_g)^{\frac{3}{2}}\sqrt{4-\lambda_i(\mathcal{L}_g)}}, & \text{if } \lambda_i \leq 2, \\ \frac{1}{\lambda_i(\mathcal{L}_g)} & \text{otherwise.}\end{cases} \tag{22}$$

Hence, for system $\mathcal{H}_\infty$ norm we have

$$||G||_\infty = \sup_\omega \max_i ||G_i(j\omega)|| = \max_i \mathcal{C}_i = \mathcal{C}_1. \tag{23}$$

Now due to the fact that in MD arrangement we have $1 \leq \lambda_1(\mathcal{L}_g) \leq 2$, and considering the fact that in this interval $\mathcal{C}_1$ takes its maximum at $\lambda_1(\mathcal{L}_g) = 1$ we have

$$||G||_\infty = \frac{2}{\lambda_1(\mathcal{L}_g)^{\frac{3}{2}}\sqrt{4 - \lambda_1(\mathcal{L}_g)}} \leq \frac{2}{\sqrt{3}}. \tag{24}$$

■

Theorems 3 and 5 show how the existence of multiple reference vehicles in a platoon can increase the robustness of the network against disturbances. In Table I, the system $\mathcal{H}_\infty$ norm of the velocity tracking and network formation dynamics on $\mathcal{P}(n, k)$ for both single and multiple reference vehicles with MD arrangement, i.e. $|\mathcal{R}| = \lceil\frac{n}{2k+1}\rceil$, is summarized.[2]

TABLE I: System $\mathcal{H}_\infty$ norm of (5) and (8) for single and multiple reference vehicles with MD arrangement.

$\lvert\mathcal{R}\rvert$	Velocity Tracking (5)	Network Formation (8)
1	$\Theta(n^2)$	$\Theta(n^3)$
$\lceil\frac{n}{2k+1}\rceil$	≤ 1	$\leq \frac{2}{\sqrt{3}}$

Based on what is presented in Table I, for the case of unit state feedback, i.e., $k_p = k_u = 1$ as mentioned in Remark 3, it is impossible to reach to a small system $\mathcal{H}_\infty$ norm with single reference vehicle and line network topology. However, via a specific selection and placement of multiple reference vehicles and appropriate design of the network, one can reach to small system norms.

B. Robustness to Time Delay

The following proposition gives a sufficient condition for which the network formation dynamics (8) remains asymptotically stable in the presence of time delay.

Proposition 5: The network formation dynamics (8) is asymptotically stable in the presence of constant time delay $\tau \in [0, \tau_{\max}]$ if $\tau_{\max} < \frac{1}{4k}$.

Proof: Based on [26], a sufficient condition for (13) to remain stable in the presence of time delay is to have

$$\tau_{\max} < \frac{1}{\rho(\mathcal{B})}, \tag{25}$$

[2]In [14] it is shown that the system $\mathcal{H}_\infty$ norm for network formation dynamics for $\mathcal{P}(n, 1)$ (line graph) is $\Theta(n^3)$, which holds for any $k < \infty$ as well. Moreover, it can be easily shown that the $\mathcal{H}_\infty$ norm of the velocity tracking dynamics is $\Theta(n^2)$, due to the fact that for line graphs we have $\lambda_1(\mathcal{L}_g) = \Theta(\frac{1}{n^2})$ [25].

where $\rho(\mathcal{B})$ is the spectral radius of $\mathcal{B}$. Applying Theorem 1 and (11), the spectral radius of $\mathcal{B}$ yields:

$$\rho(\mathcal{B}) = \frac{\lambda_{|\mathcal{F}|}(\mathcal{L}_g)}{2}\left(1+\left(1-\frac{4}{\lambda_{|\mathcal{F}|}(\mathcal{L}_g)}\right)^{\frac{1}{2}}\right), \tag{26}$$

since in the MD arrangement in which $\lambda_1(\mathcal{L}_g) \geq 1$ we have $\max_i |1-\frac{4}{\lambda_i(\mathcal{L}_g)}| = |1-\frac{4}{\lambda_{|\mathcal{F}|}(\mathcal{L}_g)}|$. Therefore, based on the upper bound on $\lambda_{|\mathcal{F}|}(\mathcal{L}_g)$ in (15), sufficient condition (25) can be rewritten as

$$\tau_{\max} < \frac{1}{d^{\mathcal{F}}_{\max} + d^{\mathcal{F}}_{\max}\left(1-\frac{2}{d^{\mathcal{F}}_{\max}}\right)^{\frac{1}{2}}} \tag{27}$$

and since $d^{\mathcal{F}}_{\max} \geq 2$ we have $d^{\mathcal{F}}_{\max} + d^{\mathcal{F}}_{\max}\left(1-\frac{2}{d^{\mathcal{F}}_{\max}}\right)^{\frac{1}{2}} \leq 2d^{\mathcal{F}}_{\max}$. This yields the sufficient condition $\tau_{\max} < \frac{1}{2d^{\mathcal{F}}_{\max}}$, and based on the fact that $d^{\mathcal{F}}_{\max} \leq 2k$ the result will be obtained. ■

Remark 6: Similar to what was mentioned in Remark 5 for velocity tracking dynamics, there is a trade-off between robustness to external disturbances and robustness to time delay for the network formation dynamics. More specifically, by increasing network connectivity k the value of $\lambda_1(\mathcal{L}_g)$ increases and based on (22) the system $\mathcal{H}_\infty$ norm decreases. On the other hand, the spectral radius of $\mathcal{B}$ increases, which results in decreasing the robustness to time delay. □

VII. SIMULATIONS

In this section, some simulation results are presented to confirm the theoretical contributions of the paper. The results are based on $\mathcal{P}(36, 4)$ and MD arrangement in which there are four reference vehicles in $\mathcal{P}(36, 4)$.

A. $\mathcal{H}_\infty$ Robustness of the Velocity Tracking and Network Formation Dynamics

In this subsection, we show how tight are the conditions provided in Theorems 3 and 5 for the $\mathcal{H}_\infty$ robustness of velocity tracking and network formation dynamics.

Figures 3 and 4 illustrates $\mathcal{H}_\infty$ norm of (8) and (5) with removing/adding a single reference vehicle from/to the MD arrangement in $\mathcal{P}(36, 4)$. Fig. 3 shows how MD arrangement of the reference vehicles introduces tight necessary and sufficient conditions for system $\mathcal{H}_\infty$ norm of (5) to be non-expansive (Theorem 3). The horizontal axis is the location of the added/removed vehicle in the platoon. In particular, if one of the four reference vehicles in the MD arrangement of $\mathcal{P}(n, k)$ is removed, the resulting $\mathcal{H}_\infty$ norm is no longer less than one. On the other hand, as can be seen from Fig. 3 if an extra reference vehicle is added to $\mathcal{P}(n, k)$ (other than the existing reference vehicles from the MD arrangement), the resulting $\mathcal{H}_\infty$ norm will be strictly less than one. The results for the same scenario are shown for the network formation dynamics (8) as shown in Fig. 4 where removing a reference vehicle makes the $\mathcal{H}_\infty$ norm of (8) larger than $\frac{2}{\sqrt{3}} \approx 1.15$. This confirms the tight sufficient condition mentioned in Theorem 5.

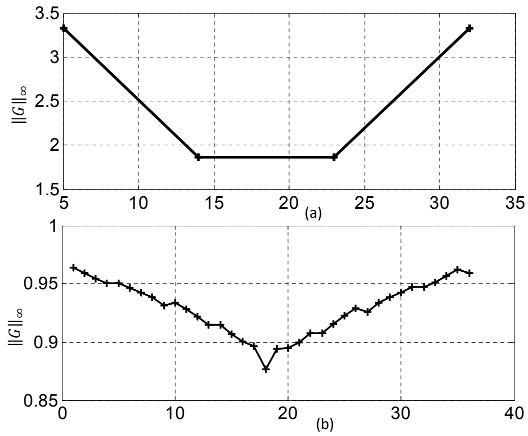

Fig. 3: $\mathcal{H}_\infty$ norm of (5) with (a) removing a Ref. vehicle and (b) adding a Ref. vehicle.

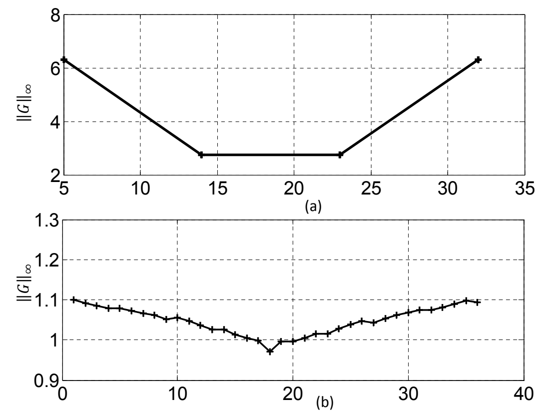

Fig. 4: $\mathcal{H}_\infty$ norm of (8) with (a) removing a Ref. vehicle and (b) adding a Ref. vehicle.

B. Effect of the Time Delay on the Stability of Velocity and Network Formation Dynamics

In this subsection, we simulate the results of Propositions 3 and 5 on the stability of velocity tracking and network formation dynamics in the presence of time delay.

Fig. 5 shows how necessary and sufficient conditions for the value of time delay mentioned in Proposition 3 apply for asymptotic stability of the velocity tracking error dynamics (5) in the presence of time delay. The sufficient condition for the stability of the network formation error dynamics (8) in the presence of time delay (presented in Proposition 5) is confirmed in Fig. 6. According to Fig. 5 and Fig. 6, a small deviation from graph theoretic bounds, proposed in Propositions 3 and 5, can deteriorate the asymptotic stability of the vehicle network dynamics.

VIII. SUMMARY AND CONCLUSIONS

In this paper a set of graph theoretic conditions for the robustness of k-nearest neighbor vehicle platoons $\mathcal{P}(n, k)$ to disturbances and time delay have been derived and analyzed. In particular, a necessary and sufficient condition for $\mathcal{P}(n, k)$ to have non-expansive $\mathcal{H}_\infty$ norm for velocity tracking dynamics has been provided (Theorem 3) by introducing a

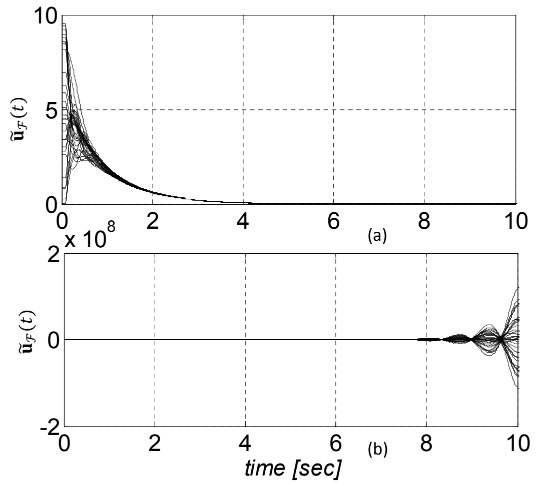

Fig. 5: (a) Velocity error in *velocity tracking* converges to zero for $\tau = 0.09 < \frac{\pi}{2k}$ and (b) diverges for $\tau = 0.4 > \frac{\pi}{8k}$.

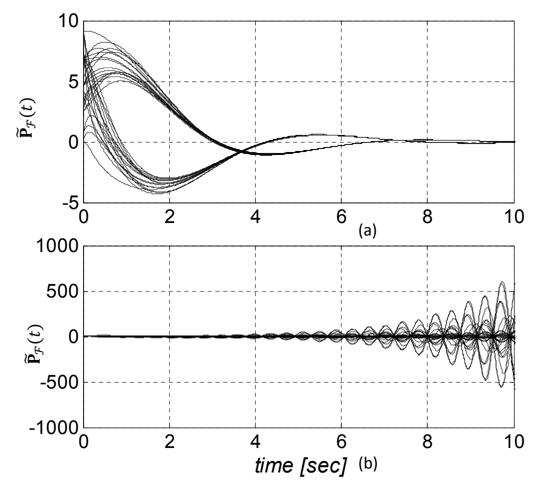

Fig. 6: (a) Position error in *network formation* converges to zero for $\tau = 0.05 < \frac{1}{4k}$ and (b) diverges for $\tau = 0.1 > \frac{1}{4k}$.

specific arrangement of reference vehicles. The effect of such arrangement of reference vehicles on $\mathcal{H}_\infty$ norm of network formation dynamics has also been investigated (Theorem 5). Furthermore, the effect of the communication delay on the stability of velocity tracking dynamics and network formation dynamics has been addressed (Propositions 3 and 5). The results show that there is a trade-off between robustness to time delay and robustness to disturbances. These results are also showing fundamental limitations in the conceptual design level of such networked control systems. An avenue for future work in this direction is to take more practical considerations and to generalize the results established in this article to directed networks with non-homogeneous control gains.

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