How much Uncertainty can be Dealt with by Feedback?

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***Abstract*—Feedback is used primarily for reducing the effects of the plant uncertainty on the performance of control systems, and as such understanding the following questions is of fundamental importance: How much uncertainty can be dealt with by feedback? What are the limitations of feedback? How does the feedback performance depend quantitatively on the system uncertainty? How can the capability of feedback be enhanced if *a priori* information about the system structure is available? As a starting point toward answering these questions, a typical class of first-order discrete-time dynamical control systems with both unknown nonlinear structure and unknown disturbances is selected for our investigation, and some concrete answers are obtained in this paper. In particular, we find that in the space of unknown nonlinear functions, the generalized Lipschitz norm is a suitable measure for characterizing the size of the structure uncertainty, and that the maximum uncertainty that can be dealt with by the feedback mechanism is described by a ball with radius $3/2+\sqrt{2}$ in this normed function space.**

***Index Terms*—Adaptive control, feedback, nonlinear, robust control, stability, stochastic systems, uncertainty.**

I. INTRODUCTION

FEEDBACK is a basic concept in automatic control. Its primary objective is to reduce the effects of the plant uncertainty on the desired control performance (e.g., stability, optimality of tracking, etc.). The uncertainty of a plant usually stems from two sources: internal (structure) uncertainty and external (disturbance) uncertainty. In general, the former is harder to cope with than the latter. The understanding of the relationship between parameter/structure uncertainty and feedback mechanism is a longstanding fundamental issue in automatic control (cf. e.g., [1]–[3]). Specific questions pointing to this issue include at least the following.

- How much uncertainty can be dealt with by feedback?
- What are the limitations of feedback?
- How does the feedback performance depend quantitatively on the plant uncertainty?
- How can the capability of feedback be enhanced if *a priori* information about the plant structure is available?

These are conundrums, on which only a few existing areas of control theory can shed some light. Robust control and adaptive control are two such areas where structure uncertainty of the plant is the main concern in the controller design.

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Robust control and its related area of robustness analysis usually require that the true plant lies in a (small) ball centered at a known nominal model and often assume that the controllers are either selected from certain given classes of systems or simply fixed (e.g., [4]). Within such a framework, substantial progress has been achieved in the understanding of the effect of uncertainties, feedback robustness, optimal robustness radii, and optimal control, via the development of various approaches and theories including H^∞ and l^1 theory, μ synthesis, small gain theorems, and gap metrics (see, e.g., [5]–[17]). The need of a nominal model with reliable model error bounds in robust control methods motivated the extensive research activities in an area called control-oriented worst case identification in the 1990s (e.g., [18]–[20]). During the same period, significant progress in linking the theories of identification, feedback, information, and complexity following the framework and philosophy developed by Zames (cf. [3], [5], [21], [22]) has also been made (see, e.g., [23]–[26]).

Adaptive control is a nonlinear feedback technique which performs identification and control simultaneously in the same feedback loop, and which is known to be a powerful tool in dealing with systems with large uncertainties. Much progress has been made in this area since the end of 1970s (cf. e.g., [27]–[30]). For linear finite-dimensional systems with uncertain parameters, a well-developed theory of adaptive control exists today, both for stochastic systems (cf. [28], [31], [32]) and for deterministic systems with small unmodeled dynamics (cf. [29]). This theory can be generalized to nonlinear systems with linear unknown parameters and with linearly growing nonlinearities (e.g., [33]). However, fundamental differences emerge between adaptive control of continuous- and discrete-time systems when one allows the nonlinearities to have a nonlinear growth rate: the design of (globally) stable adaptive control is possible only for continuous-time systems (cf. [30]), and not for discrete-time systems in general, as demonstrated rigorously in the recent works [34] and [35]. Analogously, for sampled-data control systems with uncertain nonparametric nonlinearities, it has been shown that the design of stabilizing sampled-data feedback is possible if the sampling period is small enough (cf. [36], [37]). However, if the sampling period is larger than a certain value, then globally stabilizing sampled-data feedback does not exist in general even if the nonlinearity has a linear growth rate (see [38]). The fact that sampling usually destroys many helpful properties is one of the reasons why most of the existing design methods for nonlinear control remain in the continuous-time even in the nonadaptive case (cf. [39]), albeit many results on nonlinear systems in continuous-time have their discrete-time counterparts (see, e.g., [40] and [41]).

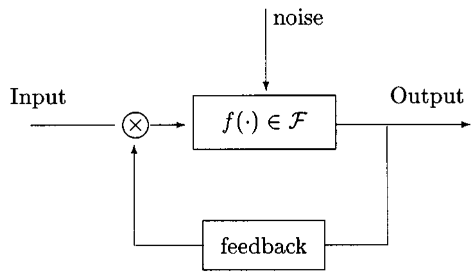

Fig. 1. Feedback control of uncertain systems.

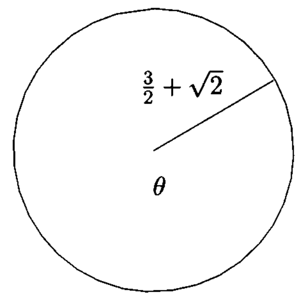

Fig. 2. The maximum uncertainty that can be dealt with by feedback is a ball in $(\mathcal{F}, \|\cdot\|)$.

Up to now, almost all of the existing results in adaptive control are not concerned with the issue of optimal robustness and are restricted to parametric models, mostly to linearly parameterized ones. Hence the understanding of the fundamental question concerning the capability and limitations of (generally defined) feedback is far from being complete, although it is hard to distinguish adaptive feedback from ordinary nonlinear feedback in general (see, e.g., [27, p. 1]).

Parametric models are of course only a special situation. The more challenging problem is to control nonparametric uncertain systems, which will be discussed in a little more detail below. Let $f(\cdot)$ be an unknown nonparametric function characterizing the nonlinear dynamics of a control system, which lies in the space of all $R^1 \to R^1$ mappings, denoted by $\mathcal{F}$ (see Fig. 1).

The traditional method is to approximate the unknown $f(\cdot)$ by certain parametric models. The existing approximation techniques (e.g., Volterra series, fuzzy and neural nets, wavelets, etc.) basically state that for x in a compact set, $f(\cdot)$ can be uniformly approximated by parametric functions of the form

$$g(\theta, x) \triangleq \sum_{i=1}^{N} a_i \sigma\left(b_i^\tau x\right)$$

where $\sigma(\cdot)$ is a known "basis" function, and a_i's and b_i's are unknown parameters or weights.

Thus, one may conceive that the above explicit parametric model $g(\theta, x)$ can be used in adaptive control instead of using the original nonparametric model $f(\cdot)$. This natural idea has some appealing features and has attracted considerable attention from researchers in recent years (e.g., [42]), but it has also several fundamental limitations/difficulties. First, in order to ensure that x (which usually represents the system state or output signals) lies in a compact set for reliable approximation, stability of the system must be established first, and the parametric model provides little (if any) help in this regard. Second, searching for the optimal parameters a_i's and b_i's usually involves in global nonlinear optimization, of which a general efficient way is still lacking by now; moreover, the on-line combination of the estimation and control (adaptive control) will further complicate the problem. Third, there always exists an approximation error in the model and hence in the control performance. Thus, it may be an advantage to consider the nonparametric model $f(\cdot)$ directly, and the nonparametric estimation methods that have been well-developed in mathematical statistics would naturally be brought to our attention. However, the statistical nonparametric estimation-based control strategy has been shown to be useful only for a class of open-loop stable systems by now (see [43] and [44]).

All of the above facts and analyzes show that, to study the questions raised at the outset, we have to place ourselves in a framework that is somewhat beyond those of the classical robust control and adaptive control. First, the system structure uncertainty may be nonlinear and nonparametric, and a useful or reliable ball containing the true plant and centered at a known nominal model may not be available *a priori*. Second, we need to study the full capability of the feedback mechanism which includes all (nonlinear and time-varying) causal mappings, and are not only restricted to a fixed feedback law or a set of specific feedback laws. We shall also work with discrete-time control models which can reflect the limitations of actuator and sensor in a certain sense when implemented with digital computers. It is fairly well known that in the present case, the high gain and nonlinear damping approaches which are very powerful in the continuous-time case are no longer effective now.

To initiate a quantitative study of the relationships between uncertainty and feedback in the framework delineated as above, we shall in this paper select a special class of first-order discrete-time dynamical control systems with matching conditions for our investigation. By introducing a suitable norm $\|\cdot\|$ (called the generalized Lipschitz norm) in the space of all nonlinear functions, we are able to give a complete characterization of the capability and limitations of the feedback mechanism for controlling this class of uncertain nonlinear systems. To be precise, we will show that:

- the maximum uncertainty that can be dealt with by feedback is a ball with radius $L = 3/2 + \sqrt{2}$ in the normed function space $(\mathcal{F}, \|\cdot\|)$, centered at the zero θ (see Fig. 2);
- if a certain "symmetric" information about the plant is available, then the above radius can be raised to $L = 4$;
- for either bounded noises or white noises, the feedback performance is bounded by a quantity reflecting the discontinuity of the plant structure.

The remainder of this paper is organized as follows. In Section II we will present the main theorems of the paper. Some auxiliary lemmas are presented in Section III, which will be used in Section IV in the proofs of the main theorems. Finally, some concluding remarks will be given in Section V.

II. Main Results

To facilitate a theoretical study, we would like in this paper to approach our problem as naked as possible, while keeping the basic nature of the problem formulation as outlined in the introduction. Thus, let us consider the following first-order discrete-time nonlinear dynamical control system:

$$y_{t+1} = f(y_t) + u_t + w_{t+1}, \qquad t \geq 0,\ y_0 \in \mathbb{R}^1 \tag{1}$$

where $\{y_t\}$ and $\{u_t\}$ are the system output and input signals, respectively. The nonlinear function $f(\cdot)$: $\mathbb{R}^1 \to \mathbb{R}^1$ is completely unknown; $\{w_t\}$ is a sequence of "unknown but bounded noises" with unknown bound $w > 0$, i.e.,

$$|w_t| \leq w, \qquad \forall\, t \geq 0. \tag{2}$$

To investigate the capability and limitations of feedback, we need to give a precise definition of it first.

Definition 2.1: A sequence $\{u_t\}$ is called a feedback control law if at each step $t \geq 0, u_t$ is a causal function of the observations $\{y_t\}$, i.e.,

$$u_t = h_t(y_0, \cdots, y_t) \tag{3}$$

where $h_t(\cdot)$: $\mathbb{R}^{t+1} \to \mathbb{R}^1$ can be an arbitrary (nonlinear and time-varying) mapping at each step t.

With the feedback mechanism defined as above, the main objective of this paper is to answer how much uncertainty in $f(\cdot)$ can be dealt with by the feedback control u_t in (1). In order to do this, we need to find a suitable measure of uncertainty first. Such a measure should be able to enable us to capture precisely the capability and limitation of feedback in dealing with structure uncertainty.

Let $\mathcal{F}$ be the space of all $\mathbb{R}^1 \to \mathbb{R}^1$ mappings, i.e., $\mathcal{F} \triangleq \{f\colon \mathbb{R}^1 \to \mathbb{R}^1\}$. Introduce a functional $\|\cdot\|$ on $\mathcal{F}$, which is defined as

$$\|f\| \triangleq \lim_{\alpha\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \frac{|f(x)-f(y)|}{|x-y|+\alpha}, \qquad \forall\, f \in \mathcal{F} \tag{4}$$

where the limit exists by the monotonicity in α.

It is easy to see that the above functional is a quasi-norm on $\mathcal{F}$, which may be called the generalized Lipschitz norm since it is closely related to the generalized Lipschitz condition as will be shown shortly. It is a true norm on the quotient space $\mathcal{F}/\theta$ defined by

$$\mathcal{F}/\theta \triangleq \{[f]\colon f \in \mathcal{F}\} \tag{5}$$

where $[f]$ signifies the equivalent class

$$[f] \triangleq \{g \in \mathcal{F}\colon \|g\| = \|f\|\} \tag{6}$$

and $\theta \triangleq \{f \in \mathcal{F}\colon \|f\| = 0\}$ is the zero in $\mathcal{F}/\theta$.

For convenience of presentation, we shall simply speak of $(\mathcal{F}, \|\cdot\|)$ as a normed linear space, and regard θ as its zero in the sequel.

For any $L > 0$ define

$$\mathcal{F}(L) \triangleq \{f \in \mathcal{F}\colon \|f\| \leq L\}. \tag{7}$$

Then $\mathcal{F}(L)$ is a ball in the space $(\mathcal{F}, \|\cdot\|)$ centered at θ with radius L.

Now, suppose the *a priori* information we have about the system (1) is that we know $f(\cdot) \in \mathcal{F}(L)$. Then L can be regarded as a measure of the size of uncertainty of our knowledge about $f(\cdot)$.

Theorem 2.1: The necessary and sufficient condition for the existence of a stabilizing feedback control law for (1) with any $f \in \mathcal{F}(L)$, is $L < 3/2 + \sqrt{2}$. To be precise, we have the following.

i) If $L < 3/2 + \sqrt{2}$, then there exists a feedback control law $h_t(\cdot), t \geq 0$ in (3) such that for any $f \in \mathcal{F}(L)$, the corresponding closed-loop control system (1) with (3) is globally stable in the sense that

$$\sup_{t\geq 0}\{|y_t| + |u_t|\} < \infty, \qquad \forall\, y_0 \in \mathbb{R}^1.$$

ii) If $L \geq 3/2 + \sqrt{2}$, then for any feedback control law $h_t(\cdot), t \geq 0$ in (3) and any $y_0 \in \mathbb{R}^1$, there always exists some $f \in \mathcal{F}(L)$ such that the corresponding closed-loop system (1) with (3) is unstable, i.e.,

$$\sup_{t\geq 0} |y_t| = \infty.$$

Remark 2.1: From Theorem 2.1, we know that $L = 3/2 + \sqrt{2}$ is a critical value of the measure of uncertainty for stabilizing (1) with $f \in \mathcal{F}(L)$. If $L < 3/2+\sqrt{2}$, then we can design a concrete feedback control law (see Theorems 2.2 and 2.3 below) which stabilizes (1) for any $f \in \mathcal{F}(L)$. But if $L \geq 3/2+\sqrt{2}$, we cannot be sure of stabilizing (1) no matter how we design the feedback control law $h_t(\cdot)$ in (3), because the plant uncertainty is too large in this case. While it is natural to consider the Lipschitz norm and the Lipschitz condition (see Remark 2.3 below) from previous studies [34], [35], [43], [44], it appears to be far from obvious why there exists a finite critical value and why this value is precisely $3/2+\sqrt{2}$. One explanation comes from our proof given in the next two sections, where it can be seen that the stabilizability of (1) hinges on the asymptotic behavior of the solutions of the following second-order linear difference equation:

$$a_{n+1} = \left(L + \frac{1}{2}\right) a_n - L a_{n-1}, \qquad n \geq 1. \tag{8}$$

To be specific, (1) is stabilizable if and only if all the solutions of (8) either converge to zero or oscillate about zero, which is precisely equivalent to the requirement $L < 3/2 + \sqrt{2}$ by [45, Th. 2.36]. Similar connections have been found previously in [34] for the critical stabilizability of an uncertain parametric model with polynomial growth nonlinearities.

Remark 2.2: It can be argued that (1) is a basic model (simplest but nontrivial) for the study of our problem. The matching condition enables us to focus our attention on uncertainty and makes it possible for us to explore the full capability of feedback in dealing with uncertainty. It also prevents the capability of feedback from being weakened at the outset by a weak control structure. While it is still necessary to investigate more general nonlinear models in future works, it may be remarked that the limitation of feedback found in Theorem 2.1 may also be regarded as a limitation of feedback in general model classes which include the class (1) as a special case.

Remark 2.3: The norm defined by (4) is closely related to the following generalized Lipschitz condition:

$$\textbf{C1)} \quad |f(x) - f(y)| \le L|x - y| + c, \qquad \forall (x, y) \in \mathbb{R}^2$$

where $L \ge 0, c \ge 0$, are constants.

It is called generalized Lipschitz condition because it includes the standard Lipschitz condition as a special case $(c = 0)$. Note that c is a quantity reflecting the possible discontinuity of the function $f(\cdot)$. Denote

$$\mathcal{F}(L, c) \triangleq \{f \in \mathcal{F}\colon f \text{ satisfies condition C1)}\}. \tag{9}$$

Obviously, $\mathcal{F}(L, c)$ is nondecreasing with respect to L and c, i.e.,

$$\mathcal{F}(L_1, c_1) \subseteq \mathcal{F}(L_2, c_2), \qquad \text{for } L_1 \le L_2, c_1 \le c_2. \tag{10}$$

Also, for the ball $\mathcal{F}(L)$ defined by (7), it is easy to prove (see Appendix A) that

$$\mathcal{F}(L, c) \subset \mathcal{F}(L), \qquad \forall\, c \ge 0 \tag{11}$$

and that for any $\gamma > 0$

$$\mathcal{F}(L) \subset \bigcup_{c \in [0,\infty)} \mathcal{F}(L + \gamma, c). \tag{12}$$

Hence for any $f \in \mathcal{F}(L)$ and any $\gamma > 0$, there exists some $c \ge 0$ such that

$$f \in \mathcal{F}(L + \gamma, c). \tag{13}$$

Next, we proceed to construct a concrete feedback law to stabilize (1) with $f \in \mathcal{F}(L)$ when $L < 3/2 + \sqrt{2}$, and at the same time to make the system outputs $\{y_t\}$ track a bounded sequence of reference signals $\{y_t^*\}$ with bound S

$$|y_t^*| \le S < \infty, \qquad t \ge 0. \tag{14}$$

Let us denote

$$\begin{aligned} \overline{b}_t &\triangleq \max_{0 \le i \le t} y_i \\ \underline{b}_t &\triangleq \min_{0 \le i \le t} y_i, \qquad t \ge 0 \end{aligned} \tag{15}$$

and

$$i_t \triangleq \arg\min_{0 \le i \le t-1} |y_t - y_i|$$

$$\text{i.e.,} \quad |y_t - y_{i_t}| = \min_{0 \le i \le t-1} |y_t - y_i|, \qquad t \ge 1. \tag{16}$$

At any time instant $t \ge 1$, the estimate of $f(y_t)$ is defined as

$$\widehat{f}_t(y_t) \triangleq y_{i_t+1} - u_{i_t} \tag{17}$$

which can be rewritten as

$$\widehat{f}_t(y_t) = f(y_{i_t}) + w_{i_t+1}, \qquad t \ge 1. \tag{18}$$

We remark that the estimator (16), (17) may be refered to as the nearest neighbor (NN) estimator for $f(\cdot)$ (cf. e.g., [46], [47]), as can be seen intuitively from (18). It is a natural one when we only know the generalized Lipschitz continuity of $f(\cdot)$ and the boundedness of the noises $\{w_t\}$. Better estimators may be constructed if more information about $f(\cdot)$ or $\{w_t\}$ is available, as will be shown later in (31) or (26).

Denote

$$u_t' \triangleq -\widehat{f}_t(y_t) + \tfrac{1}{2}(\underline{b}_t + \overline{b}_t), \qquad t \ge 1 \tag{19}$$

$$u_t'' \triangleq -\widehat{f}_t(y_t) + y_{t+1}^*, \qquad t \ge 1. \tag{20}$$

Then the feedback control law is defined as $u_0 \triangleq 0$

$$u_t \triangleq \begin{cases} u_t', & \text{if } |y_t - y_{i_t}| > \epsilon \\ u_t'', & \text{if } |y_t - y_{i_t}| \le \epsilon \end{cases} \quad t \ge 1 \tag{21}$$

where $\epsilon > 0$ can be chosen arbitrarily.

Theorem 2.2: For any $f \in \mathcal{F}(L)$ with $L < 3/2 + \sqrt{2}$, the feedback control (15)–(21) globally stabilizes the corresponding system (1) with the following tracking performance:

$$\overline{\lim_{t\to\infty}} |y_t - y_t^*| \le c + 2w$$

where c is defined in (13) with $\gamma > 0$ chosen to satisfy $L_1 \triangleq L + \gamma < 3/2 + \sqrt{2}$, and w is defined in (2).

Remark 2.4: In the feedback law (21), u_t' is designed mainly for the purpose of stabilizing the system, and u_t'' for making the output $\{y_{t+1}\}$ track $\{y_{t+1}^*\}$ when the estimation (prediction) of $f(y_t)$ is good enough in the sense that $|y_t - y_{i_t}| \le \epsilon$. However, when $|y_t - y_{i_t}| > \epsilon$, we are not sure of the goodness of the NN estimator (17) for $f(y_t)$, and the stability issue becomes the main concern. In this case, one will at least expect in the next step to have y_{t+1} not far from the past outputs $\{y_0, \cdots, y_t\}$, as the accuracy of the NN estimate of $f(y_{t+1})$ depends on the distance $|y_{t+1} - y_{i_{t+1}}|$. So conservatively, the best way is trying to place y_{t+1} at the center of the past outputs as in (19). In (21), ϵ is a design parameter, which has no effect on the asymptotic tracking performance bound as can be seen from Theorem 2.2. This attributes to the fact that u_t defined by (21) will be identical to u_t'' of (20) after some finite time, as long as the system signals are bounded (see Lemma 3.4 in the next section). However, larger ϵ may cause larger variations in the transient response; while smaller ϵ may make the transient response time longer.

Remark 2.5: The NN estimator-based stabilizing feedback (15)–(21) appears to require infinite memory for implementation, which is not a desirable property from a practical point of view. However, from a theoretical point of view, our results show that at least the observed data carries enough information about the uncertain function $f(\cdot)$ to be able to stabilize the system. Moreover, thanks to the robustness of the controller (15)–(21) with respect to bounded noises $\{w_t\}$ (and hence to bounded estimation errors), it is possible to construct an easily implementable controller to approximate (15)–(21) by making a tradeoff between computational complexity and performance accuracy. One way of doing this is to divide the output space into sufficiently small intervals (depending on the performance requirements), and on each of which keep only one datum and discard others. This would only result in bounded estimation errors, but at the same time significantly reduces the demand on memory for implementation.

As one would expect, if we further assume that the disturbance $\{w_t\}$ is a white noise sequence, then the tracking error bound can be improved. To this end, we next assume that $\{w_t, t \geq 0\}$ is a martingale difference sequence, i.e.,

$$E[w_{t+1} \,|\, \mathcal{B}_t] = 0, \qquad t \geq 0 \tag{22}$$

where $\mathcal{B}_t \triangleq \sigma\{w_0, \cdots, w_t\}$ is the σ-algebra generated by $\{w_0, \cdots, w_t\}$. In order to make use of the property of martingales, we next change the estimate defined by (17) into some averaging form.

We first introduce some notations. Define

$$\delta_j \triangleq [j\epsilon, (j+1)\epsilon)], \qquad j \in Z \triangleq \{\text{all integers}\} \tag{23}$$

where ϵ is the same as in (21). Then

$$\bigcup_{j\in Z} \delta_j = (-\infty, +\infty)$$

and

$$\delta_i \bigcap \delta_j = \emptyset \text{ (null set)}, \qquad \text{for } i \neq j.$$

For any $y \in \mathbb{R}^1$, define the interval-valued function $\triangle(\cdot)$ as

$$\triangle(y) \triangleq \delta_{j-1} \bigcup \delta_j \bigcup \delta_{j+1} \qquad \text{if } y \in \delta_j,\ j \in Z. \tag{24}$$

Intuitively speaking, $\triangle(y)$ covers the "ϵ-neighborhood" of y.

If for some $t \geq 1, |y_t - y_{i_t}| \leq \epsilon$, then by the definitions above, we have

$$\sum_{i=0}^{t-1} I_{\triangle(y_t)}(y_i) > 0 \tag{25}$$

where, for any $A \subset \mathbb{R}^1, I_A(\cdot)$ is the indicator function

$$I_A(x) \triangleq \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

Hence when $|y_t - y_{i_t}| \leq \epsilon$, we may define the estimate of $f(y_t)$ as

$$\overline{f}_t(y_t) \triangleq \frac{\sum_{i=0}^{t-1} (y_{i+1} - u_i) I_{\triangle(y_t)}(y_i)}{\sum_{i=0}^{t-1} I_{\triangle(y_t)}(y_i)} \tag{26}$$

and correspondingly change (20) into

$$u_t'' \triangleq -\overline{f}_t(y_t) + y_{t+1}^*, \qquad t \geq 1. \tag{27}$$

We have the following theorem.

Theorem 2.3: Let $\{w_t\}$ be a bounded martingale difference sequence. Then, for any $f \in \mathcal{F}(L)$ with $L < 3/2 + \sqrt{2}$, the corresponding closed-loop system defined by (1), (15)–(19), (26)–(27), and (21) has the following tracking performance bound:

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=1}^{T} (y_t - y_t^* - w_t)^2 \leq (2L_1\epsilon + c)^2 \tag{28}$$

where c is defined in (13) with any $\gamma > 0$ satisfying $L_1 \triangleq L + \gamma < 3/2 + \sqrt{2}$, and ϵ is defined in (21).

Remark 2.6: From an application point of view, it is also important to be able to verify or falsify the condition $f \in \mathcal{F}(L)$ with $L < 3/2 + \sqrt{2}$. We note that for this task it is not necessary to require the full knowledge of the functional representation, and that most of the practical systems are not really "black boxes"—some information, more or less, should be available *a priori*. An extreme yet common example is the class of bounded functions, which we know will satisfy the above condition without any knowledge of the functional representation (the Lipschitz norms are all zero for this class of functions). Although the stability issue is trivial in this case, the feedback laws of Theorems 2.2 and 3.3 are still valuable as they lead to nontrivial closed-loop tracking performance bounds.

As one would also expect, if we have additional restriction (or *a priori* information) on $f(\cdot)$, then the critical value of L would increase. As an example illustrating this, we suppose that the value of $f(-x)$ would be known if we know the value of $f(x)$ for any $x \in \mathbb{R}^1$. Typical examples include functions like $f(x) = f(-x)$, and $f(x) = -f(-x)$, etc. For simplicity, we only consider the case $f(x) = f(-x)$ in the sequel. Other cases can be treated analogously. In this case, the system uncertainty set $\mathcal{F}(L)$ will shrink to

$$\mathcal{G}(L) \triangleq \{f \in \mathcal{F}(L)\colon f(x) = f(-x), \forall\, x \in \mathbb{R}^1\}. \tag{29}$$

Similar to Theorem 2.1, we have the following result.

Theorem 2.4: The necessary and sufficient condition for the existence of a stabilizing feedback control law for (1) with arbitrary $f \in \mathcal{G}(L)$, is $L < 4$. To be precise, we have the following.

i) If $L < 4$, then there exists a feedback control law $h_t(\cdot), t \geq 0$ in (3) such that for any $f \in \mathcal{G}(L)$, the corresponding closed-loop control system (1) with (3) is globally stable in the sense that

$$\sup_{t\geq 0}\{|y_t| + |u_t|\} < \infty, \qquad \forall\, y_0 \in \mathbb{R}^1.$$

ii) If $L \geq 4$, then for any feedback control law $h_t(\cdot), t \geq 0$ in (3) and any $y_0 \in \mathbb{R}^1$, there always exists some $f \in \mathcal{G}(L)$ such that the corresponding closed-loop system (1) with (3) is unstable, i.e.,

$$\sup_{t\geq 0} |y_t| = \infty.$$

Theorem 2.4 shows that the critical value of the capability of feedback for stabilizing unknown systems lying in the set $\mathcal{G}(L)$ is $L = 4$, higher than $L = 3/2 + \sqrt{2} \approx 2.914$ for $\mathcal{F}(L)$, thanks to the additional condition (or *a priori* information) on the unknown function $f(\cdot)$.

Similar to the previous case, we can also design a concrete feedback law to stabilize (1) with $f \in \mathcal{G}(L), L < 4$. Again, our objective is to make the system outputs $\{y_t\}$ track a desired bounded sequence of reference signals $\{y_t^*\}$ which satisfies (14).

For (1) with $f \in \mathcal{G}(L)$, because of the "symmetry" of the information provided by (29), we replace i_t defined in (16) by

$$j_t \triangleq \arg\min_{0\le i\le t-1} \left||y_t| - |y_i|\right|, \qquad t \ge 1. \tag{30}$$

We still adopt the estimate of $f(y_t)$ defined by (17), i.e.,

$$\hat{\hat{f}}_t(y_t) \triangleq y_{j_t+1} - u_{j_t} \tag{31}$$

but here j_t is defined by (30).

If we define the control law as $u_0 = 0$

$$u_t = -\hat{\hat{f}}_t(y_t) + y^*_{t+1}, \qquad t \ge 1 \tag{32}$$

then it can be shown that this feedback law is globally stabilizing and the tracking error bound is the same as that in Theorem 2.2 [see the proof of Theorem 2.4 i) in Section IV].

III. Some Auxiliary Lemmas

In this section, we present some auxiliary lemmas which will be needed in the proofs of the main theorems stated in the last section.

Lemma 3.1: Let $L \in (0, 3/2 + \sqrt{2})$ and $d \ge 0$ be two constants. If a sequence $\{a_n, n \ge 0\}$ satisfies

$$a_{n+1} \le L(a_n - a_{n-1}) + \tfrac{1}{2}a_n + d, \qquad n \ge 1 \tag{33}$$

with $a_0 = 0$ and $a_1 = 1$, then there exists some $d_0 > 0$, such that if $d \in [0, d_0]$, there exists some $N \ge 1$ such that

$$a_n \ge a_{n-1}, \quad 1 \le n \le N \quad \text{and} \quad a_{N+1} < a_N. \tag{34}$$

Proof: We adopt the contradiction argument and follow some proof ideas in oscillation theory (cf. [45]). Suppose that

$$a_n \ge a_{n-1}, \qquad \forall n \ge 1. \tag{35}$$

Then it is obvious that $a_n \ge 1, \forall n \ge 1$ and $x_n \triangleq a_n/a_{n-1} \ge 1$. Hence, dividing both sides of (33) by a_n, we have

$$x_{n+1} \le L\left(1 - \frac{1}{x_n}\right) + \frac{1}{2} + d.$$

Now, if we denote $b \triangleq \underline{\lim}_{n\to\infty} x_n \ge 1$, then we have

$$b \le L\left(1 - \frac{1}{b}\right) + \frac{1}{2} + d.$$

From this, it is easy to see that $b \ne 1$ provided that $d \in [0, 1/2)$. Futhermore, the above inequality can be rewritten as

$$L \ge \frac{b^2 - \frac{1}{2}b - db}{b-1}. \tag{36}$$

Now, by the elementary method in calculus, it can be shown that

$$\min_{1<b<\infty} \frac{b^2 - \frac{1}{2}b}{b-1} = \frac{3}{2} + \sqrt{2}.$$

Hence, for any $\epsilon > 0$ there exists a constant $d_0 \in (0, 1/2)$ small enough such that whenever $d \in [0, d_0]$

$$\min_{1<b<\infty} \frac{b^2 - \frac{1}{2}b - db}{b-1} > \frac{3}{2} + \sqrt{2} - \epsilon.$$

Consequently, by (36)

$$L > \tfrac{3}{2} + \sqrt{2} - \epsilon.$$

From this we will get a contradiction if we take ϵ to be any (fixed) value in the interval $(0, 3/2 + \sqrt{2} - L)$. Hence, (35) is not true and the proof of the lemma is completed. □

Lemma 3.2: Let $L \ge 3/2 + \sqrt{2}$ be a constant. If a sequence $\{a_n, n \ge 0\}$ satisfies

$$a_{n+1} \ge L(a_n - a_{n-1}) + \tfrac{1}{2}a_n, \qquad n \ge 1 \tag{37}$$

with $a_0 = 0, a_1 \ge 1$, then $\{a_n\}$ is strictly increasing and tends to infinity, i.e.,

$$a_{n+1} - a_n > 0, \qquad \forall n \ge 1 \quad \text{and} \quad \lim_{n\to\infty} a_n = \infty. \tag{38}$$

Moreover

$$L(a_n - a_{n-1}) - \tfrac{1}{2}a_n > 0, \qquad \forall n \ge 1. \tag{39}$$

Proof: Since $\triangle \triangleq (L + (1/2))^2 - 4L \ge 0$ for $L \ge 3/2 + \sqrt{2}$, the following quadratic equation:

$$x^2 - \left(L + \tfrac{1}{2}\right)x + L = 0$$

has two real roots, which are denoted by λ_1 and λ_2. Then we have

$$\lambda_1 + \lambda_2 = L + \tfrac{1}{2} \tag{40}$$

$$\lambda_1 \cdot \lambda_2 = L. \tag{41}$$

From (40) and (41), it is clear that $\lambda_1 > 0$ and $\lambda_2 > 0$. Now we prove that $\lambda_1 > 1$ and $\lambda_2 > 1$. Otherwise, if, for example, $\lambda_1 \le 1$, then $\lambda_2 \ge L$ by (41). Hence by (40), $\lambda_1 \le 1/2$. Consequently, by (40) again and $L \ge 3/2 + \sqrt{2}$, we have

$$\lambda_1 \cdot \lambda_2 \le \tfrac{1}{2}\lambda_2 < \tfrac{1}{2}\left(L + \tfrac{1}{2}\right) = \tfrac{1}{2}L + \tfrac{1}{4} < L$$

which contradicts to (41). Hence $\lambda_1 > 1$ and $\lambda_2 > 1$.

By (40) and (41) we rewrite (37) as

$$(a_{n+1} - \lambda_1 a_n) \ge \lambda_2(a_n - \lambda_1 a_{n-1}), \qquad n \ge 1.$$

Iterating this inequality gives

$$(a_{n+1} - \lambda_1 a_n) \ge \lambda_2^n(a_1 - \lambda_1 a_0) \ge \lambda_2^n, \qquad n \ge 1.$$

So by $\lambda_2 > 1$, we have $\forall n \ge 1$

$$a_{n+1} - \lambda_1 a_n > 0 \quad \text{and} \quad (a_n - \lambda_1 a_{n-1}) \underset{n\to\infty}{\longrightarrow} \infty. \tag{42}$$

Hence by $\lambda_1 > 1$, (38) holds.

Now by the symmetry of λ_1 and λ_2 in (40), we may assume $\lambda_1 \ge 1/2(L + (1/2))$. Then by (42) and the fact that

$$\frac{1}{2}\left(L + \frac{1}{2}\right) > \frac{L}{L - 1/2}, \qquad \text{for } L \ge \frac{3}{2} + \sqrt{2}$$

we have

$$\begin{aligned} L(a_n - a_{n-1}) - \frac{1}{2} a_n &= \left(L - \frac{1}{2}\right) a_n - L a_{n-1} \\ &= \left(L - \frac{1}{2}\right)\left(a_n - \frac{L}{L-1/2} a_{n-1}\right) \\ &\geq \left(L - \frac{1}{2}\right)(a_n - \lambda_1 a_{n-1}) > 0 \end{aligned}$$

which is (39). Hence the proof of Lemma 3.2 is completed. □

Lemma 3.3: Let $L \in (0, 3/2 + \sqrt{2}), d \geq 0$ and $n_0 \geq 0$ be constants. If a nonnegative sequence $\{h_n, n \geq 0\}$ satisfies

$$h_{n+1} \leq \left(L \max_{0\leq i\leq n} h_i - \frac{1}{2}\sum_{i=0}^{n} h_i + d\right)^+, \qquad \forall n \geq n_0 \quad (43)$$

where $(x)^+ \triangleq \max\{x, 0\}, \forall x \in \mathbb{R}^1$, then

$$\lim_{n\to\infty} \sum_{i=0}^{n} h_i < \infty. \quad (44)$$

Proof: We adopt the contradiction argument. Suppose that

$$\sum_{i=0}^{n} h_i \to \infty. \quad (45)$$

We first show that

$$L \max_{0\leq i\leq n} h_i - \frac{1}{2}\sum_{i=0}^{n} h_i + d > 0, \qquad \forall n \geq n_0. \quad (46)$$

Actually, if (46) were not hold for some $n_1 \geq n_0$, then by (43) we would have $h_{n_1+1} = 0$. Hence (46) does not hold for n_1+1 too. Repeating this argument, we see that (46) does not hold for any $n \geq n_1$ and that $h_n = 0$ for any $n \geq n_1+1$. This contradicts our supposition (45). Hence (46) holds.

Now, by (46), we can rewrite (43) as

$$h_{n+1} \leq L \max_{0\leq i\leq n} h_i - \frac{1}{2}\sum_{i=0}^{n} h_i + d, \qquad \forall n \geq n_0. \quad (47)$$

By (45), (47), and $h_n \geq 0$, we know that $\max_{0\leq i\leq n} h_i \to \infty$. Hence, there exists some $n_1 > n_0$ such that

$$h_{n_1} > \max_{0\leq i\leq n_1-1} h_i \quad \text{and} \quad h_{n_1} \geq \frac{d}{d_0} \quad (48)$$

where d_0 is defined in Lemma 3.1. Moreover, we can choose a strictly increasing subsequence $\{h_{n_j}, j \geq 2\}$ from $\{h_n, n > n_1\}$ such that $h_{n_{j+1}} > h_{n_j}$ and

$$h_n \leq h_{n_j}, \qquad \forall n_j \leq n < n_{j+1},\ j \geq 1. \quad (49)$$

Then by (47)–(49), we have for $j \geq 1$

$$\begin{aligned} h_{n_{j+1}} &\leq L \max_{0\leq i\leq n_{j+1}-1} h_i - \frac{1}{2}\sum_{i=0}^{n_{j+1}-1} h_i + d \\ &\leq L h_{n_j} - \frac{1}{2}\sum_{k=1}^{j} h_{n_k} + d. \end{aligned} \quad (50)$$

Let $a_0 \triangleq 0$, and $a_j \triangleq (1/h_{n_1})\sum_{k=1}^{j} h_{n_k} j \geq 1$, then $a_{j+1} - a_j = (1/h_{n_1}) h_{n_{j+1}}$ and hence by (50)

$$a_{j+1} - a_j \leq L(a_j - a_{j-1}) - \frac{1}{2} a_j + \frac{d}{h_{n_1}}, \qquad j \geq 1.$$

Thus, we have $a_0 = 0, a_1 = 1$, and

$$a_{j+1} \leq L(a_j - a_{j-1}) + \frac{1}{2} a_j + \frac{d}{h_{n_1}}, \quad j \geq 1.$$

From this, by Lemma 3.1 and (48), we have for some $J \geq 1, a_{J+1} < a_J$. Then we must have

$$h_{n_{J+1}} < 0 \quad (51)$$

which contradicts $h_n \geq 0, \forall n \geq 0$.

Hence the supposition (45) is incorrect and Lemma 3.3 holds. □

Lemma 3.4: If a sequence $\{z_n, n \geq 0\}$ is bounded, i.e., $|z_n| \leq M < \infty, \forall n \geq 0$, then

$$\overline{\lim_{n\to\infty}} |z_n - z_{i_n}| = 0 \quad (52)$$

where

$$i_n \triangleq \arg\min_{0\leq i\leq n-1} |z_n - z_i|. \quad (53)$$

Proof: We adopt the contradiction argument. Suppose that

$$\overline{\lim_{n\to\infty}} |z_n - z_{i_n}| = \varepsilon > 0. \quad (54)$$

Then there exists a subsequence $\{z_{n_j}, j \geq 1\}$ such that

$$\left|z_{n_j} - z_{i_{(n_j)}}\right| > \frac{\varepsilon}{2}, \qquad \forall j \geq 1.$$

So by definition (53), we have

$$\left|z_{n_j} - z_i\right| > \frac{\varepsilon}{2}, \qquad \forall 0 \leq i < n_j.$$

Therefore $|z_{n_j} - z_{n_k}| > \varepsilon/2, \forall k < j$, or for any $z_{n_j}, z_{n_k}, k \neq j$, we have $|z_{n_j} - z_{n_k}| > \varepsilon/2$, which means that $\{z_{n_j}, j \geq 1\}$ has no convergence points, and hence obviously contradicts to $|z_{n_j}| \leq M, \forall j \geq 1$. Therefore (54) is incorrect and Lemma 3.4 holds. □

The above four lemmas will be used in the proofs of Theorems 2.1 and 2.2, while the following three lemmas are needed in the proof of Theorem 2.4. Since the proof ideas of the following Lemmas 3.5–3.7 are similar to those given above, we place the proof details in Appendix B.

Lemma 3.5: Let $0 < L < 4, d \geq 0$. If a sequence $\{a_n, n \geq 0\}$ satisfies

$$a_{n+1} \leq L(a_n - a_{n-1}) + d, \qquad n \geq 1 \quad (55)$$

with $a_0 = 0$ and $a_1 = 1$, then there exists some $d_0 > 0$, such that for any $d \in [0, d_0]$, there exists some $N \geq 1$ such that

$$a_n \geq a_{n-1}, \qquad 1 \leq n \leq N \quad \text{and} \quad a_{N+1} < a_N. \quad (56)$$

Lemma 3.6: Let $L \geq 4$. If a sequence $\{a_n, n \geq 0\}$ satisfies

$$a_{n+1} \geq L(a_n - a_{n-1}), \qquad n \geq 1 \quad (57)$$

with $a_0 \le 1, a_1 \ge L+1$, then $\{a_n\}$ is strictly increasing and tends to infinity, i.e.,

$$a_{n+1} - a_n > 0, \qquad \forall n \ge 1 \quad \text{and} \quad \lim_{n\to\infty} a_n = \infty \tag{58}$$

and moreover

$$L(a_n - a_{n-1}) - a_n > 0, \qquad \forall n \ge 1. \tag{59}$$

Lemma 3.7: Let $0 < L < 4, d \ge 0$, and $\{z_n, n \ge 0\}$ be any bounded sequence. If a nonnegative sequence $\{h_n, n \ge 0\}$ satisfies

$$|h_{n+1} - z_{n+1}| \le L \min_{0\le i\le n-1} |h_n - h_i| + d, \qquad n \ge 1 \tag{60}$$

with any $h_0, h_1 \ge 0$, then

$$\overline{\lim_{n\to\infty}} |h_n - z_n| \le d. \tag{61}$$

IV. Proofs of the Theorems

Proof of Theorem 2.1

First, we introduce some notation, which will also be used in the proof of Theorem 2.2. Denote

$$B_t \triangleq [\underline{b}_t, \overline{b}_t], \qquad \triangle B_t \triangleq B_t - B_{t-1} \tag{62}$$

and

$$|B_t| \triangleq \overline{b}_t - \underline{b}_t \quad |\triangle B_t| \triangleq |B_t| - |B_{t-1}| \tag{63}$$

where $\Delta B_0 \triangleq B_0, \underline{b}_t$ and $\overline{b}_t$ are defined in (15). Since by the definition (15)

$$\overline{b}_t \ge \overline{b}_{t-1}, \quad \underline{b}_t \le \underline{b}_{t-1} \quad \text{and} \quad (\overline{b}_t - \overline{b}_{t-1})(\underline{b}_t - \underline{b}_{t-1}) = 0$$

we know that the interval sequence $\{B_t, t \ge 0\}$ is nondecreasing and that $\triangle B_t$ is also an interval (can be a null set $\emptyset$) and

$$B_t = \bigcup_{i=0}^{t} \triangle B_i \quad \text{and} \quad \triangle B_i \bigcap \triangle B_j = \emptyset, \qquad i \ne j. \tag{64}$$

For any point $a \in \mathbb{R}^1$ and any set $B \subset \mathbb{R}^1$, define a distance function $d(\cdot,\cdot)$ as

$$d(a, B) \triangleq \inf_{b\in B} |a - b| \tag{65}$$

and if $B = \{b\}$, we rewrite $d(a, B)$ as $d(a, b) \triangleq |a - b|$.

Then it is clear that $|\triangle B_t| = d(y_t, B_{t-1}), t \ge 1$.

The first conclusion i) follows naturally from Theorem 2.2 whose proof will be given later. Here we only give the proof for the second conclusion ii).

We will show that if $L \ge 3/2 + \sqrt{2}$, then for any given feedback control law $\{u_t\}$, there always exists some $f \in \mathcal{F}(L, 0) \subset \mathcal{F}(L)$ such that the corresponding closed-loop system (1) with this $\{u_t\}$ is unstable.

For the sake of convenience, the standard Lipschitz condition is stated explicitly as

$$\textbf{C2)} \quad |f(x) - f(y)| \le L|x - y|, \qquad \forall x, y \in \mathbb{R}^1.$$

By (9), we have

$$\mathcal{F}(L, 0) = \{f\colon f \text{ satisfies condition C2)}\}. \tag{66}$$

We divide the following proof into four steps.

Step 1: First, for any y_0 and $f \in \mathcal{F}(L, 0), f(y_0)$ could be any value on $(-\infty, +\infty)$, so we have

$$\left.\begin{array}{l} \mathcal{F}_0' \triangleq \{f\colon f(y_0) = 1; f \in \mathcal{F}(L, 0)\} \ne \emptyset \\ \mathcal{F}_0'' \triangleq \{f\colon f(y_0) = -1; f \in \mathcal{F}(L, 0)\} \ne \emptyset \end{array}\right\} \tag{67}$$

where $\emptyset$ denotes the null set. Then, for any $f' \in \mathcal{F}_0', f'' \in \mathcal{F}_0''$, we have

$$f'(y_0) = 1 \quad \text{and} \quad f''(y_0) = -1. \tag{68}$$

Then, $|f'(y_0) - f''(y_0)| = 2$. Hence, for any $u_0 = h_0(y_0) \in \mathbb{R}^1$ and any $w_1 \in \mathbb{R}^1$

$$d(f'(y_0) + u_0 + w_1, f''(y_0) + u_0 + w_1) = 2.$$

From this, it is obvious that

$$\begin{aligned} \max\{&d(f'(y_0) + u_0 + w_1, y_0), \\ &d(f''(y_0) + u_0 + w_1, y_0)\} \ge 1. \end{aligned} \tag{69}$$

Now define

$$\mathcal{F}_0 \triangleq \begin{cases} \mathcal{F}_0', & \text{if } d(f'(y_0) + u_0 + w_1, y_0) \ge 1 \\ \mathcal{F}_0'', & \text{otherwise.} \end{cases}$$

By (67), (68), we have $\mathcal{F}_0 \ne \emptyset$ and for all $f \in \mathcal{F}_0$

$$f(y_0) = \text{const. (either 1 or} - 1). \tag{70}$$

Also for any $f \in \mathcal{F}_0$, by (1) and (69) we have

$$\begin{aligned} |\triangle B_1| &= |B_1| = d(y_1, y_0) \\ &= d(f(y_0) + u_0 + w_1, y_0) \ge 1 \end{aligned} \tag{71}$$

where $|B_1|$ and $|\triangle B_1|$ are defined as in (63).

Obviously, $|\triangle B_1| = |y_1 - y_0| = |f(y_0) + u_0 + w_1 - y_0|$ is constant for any $f \in \mathcal{F}_0$.

Let $b_0 \triangleq y_0$. Then $x_0 \triangleq f(b_0) = f(y_0)$ is constant for all $f \in \mathcal{F}_0$.

Under condition C2), for $f \in \mathcal{F}_0, f(y_1)$ could be any value in the following interval by (71):

$$\begin{aligned} &[f(y_0) - L|\triangle B_1|, f(y_0) + L|\triangle B_1|] \\ &= [x_0 - L|\triangle B_1|, x_0 + L|\triangle B_1|]. \end{aligned}$$

Step 2: Next, we define

$$\left.\begin{array}{l} \mathcal{F}_1' \triangleq \{f\colon f(y_1) = x_0 + L|\triangle B_1|; f \in \mathcal{F}_0\} \ne \emptyset \\ \mathcal{F}_1'' \triangleq \{f\colon f(y_1) = x_0 - L|\triangle B_1|; f \in \mathcal{F}_0\} \ne \emptyset. \end{array}\right\} \tag{72}$$

For any $f' \in \mathcal{F}_1'$ and $f'' \in \mathcal{F}_1''$, we have

$$f'(y_1) = x_0 + L|\triangle B_1| \quad \text{and} \quad f''(y_1) = x_0 - L|\triangle B_1|. \tag{73}$$

Then $|f'(y_1) - f''(y_1)| = 2L|\triangle B_1|$. Hence, for any $u_1 = h_1(y_0, y_1) \in \mathbb{R}^1$ and any $w_2 \in \mathbb{R}^1$

$$|(f'(y_1) + u_1 + w_2) - (f''(y_1) + u_1 + w_2)| = 2L|\triangle B_1|.$$

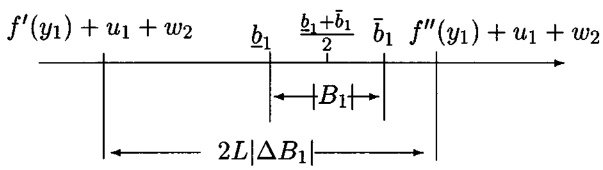

Fig. 3. Illustration of (74).

From this, it is obvious that

$$\max\{d(f'(y_1)+u_1+w_2,\tfrac{1}{2}(\underline{b}_1+\bar{b}_1)),\ d(f''(y_1)+u_1+w_2,\tfrac{1}{2}(\underline{b}_1+\bar{b}_1))\} \geq L|\triangle B_1|$$

where $\underline{b}_1, \bar{b}_1$ are defined in (15). Furthermore, by the definition (65), we have (see, e.g., Fig. 3)

$$\max\{d(f'(y_1)+u_1+w_2,B_1), d(f''(y_1)+u_1+w_2,B_1)\} \geq L|\triangle B_1| - \tfrac{1}{2}|B_1|. \qquad (74)$$

Now define

$$\mathcal{F}_1 \triangleq \begin{cases} \mathcal{F}_1', & \text{if } d(f'(y_1)+u_1+w_2,B_1) \geq L|\triangle B_1| - \frac{1}{2}|B_1|; \\ \mathcal{F}_1'', & \text{otherwise.} \end{cases}$$

It follows from (72), (73) that $\mathcal{F}_1 \neq \emptyset$ and for all $f \in \mathcal{F}_1$

$$f(y_1) = \text{const.} \qquad (75)$$

Moreover, for any $f \in \mathcal{F}_1$, by (1) and (74) we have

$$|\triangle B_2| = d(y_2,B_1) = d(f(y_1)+u_1+w_2,B_1) \geq L|\triangle B_1| - \tfrac{1}{2}|B_1| > 0 \qquad (76)$$

where the last inequality follows from $|\triangle B_1| = |B_1|$ and $L \geq 3/2+\sqrt{2}$, and where $|\triangle B_2|$ is defined as in (63). By (75) and (76), $|\triangle B_2|$ is constant for all $f \in \mathcal{F}_1$.

Since $|\triangle B_2| > 0$, we have $y_2 \notin B_1 = [\underline{b}_1, \bar{b}_1]$. Now, denote

$$b_1 \triangleq \begin{cases} \bar{b}_1, & \text{if } y_2 > \bar{b}_1 \\ \underline{b}_1, & \text{if } y_2 < \underline{b}_1. \end{cases} \qquad (77)$$

we then have $|\triangle B_2| = |y_2 - b_1|$.

Let $x_1 \triangleq f(b_1)$. By (70), (75), and $\mathcal{F}_1 \subset \mathcal{F}_0$, we know that $f(y_0)$ and $f(y_1)$ are constant for all $f \in \mathcal{F}_1$. Hence by the definitions of $b_1, \bar{b}_1$, and $\underline{b}_1$, we know that x_1 is also constant for all $f \in \mathcal{F}_1$.

Consequently, for any $f \in \mathcal{F}_1$, under condition C2), $f(y_2)$ could be any value in the interval

$$[f(b_1) - L|\triangle B_2|, f(b_1) + L|\triangle B_2|] = [x_1 - L|\triangle B_2|, x_1 + L|\triangle B_2|].$$

Step 3: Continue the above procedure and recursively define for $i \geq 2$

$$\left.\begin{aligned} \mathcal{F}_i' &\triangleq \{f\colon f(y_i) = x_{i-1} + L|\triangle B_i|; f \in \mathcal{F}_{i-1}\} \\ \mathcal{F}_i'' &\triangleq \{f\colon f(y_i) = x_{i-1} - L|\triangle B_i|; f \in \mathcal{F}_{i-1}\} \end{aligned}\right\}$$

with $x_{i-1} \triangleq f(b_{i-1})$ and $|\triangle B_i|$ being constants for all $f \in \mathcal{F}_{i-1}$. Also, define

$$\mathcal{F}_i \triangleq \begin{cases} \mathcal{F}_i', & \text{if } d(f'(y_i)+u_i+w_{i+1},B_i) \geq L|\triangle B_i| - \frac{1}{2}|B_i| \\ \mathcal{F}_i'', & \text{otherwise.} \end{cases}$$

Similar to (75) and (76), we know that for any $f \in \mathcal{F}_i, f(y_i)$ is constant and

$$|\triangle B_{i+1}| = d(y_{i+1},B_i) = d(f(y_i)+u_i+w_{i+1},B_i) \geq L|\triangle B_i| - \tfrac{1}{2}|B_i| > 0, \qquad i \geq 2 \qquad (78)$$

where the last inequality follows from Lemma 3.2 [by setting $a_i = |B_i|$ in (39)].

Since $|\triangle B_{i+1}| > 0$, we have $y_{i+1} \notin B_i = [\underline{b}_i, \bar{b}_i]$. Similar to (77), we denote

$$b_i \triangleq \begin{cases} \bar{b}_i, & \text{if } y_{i+1} > \bar{b}_i \\ \underline{b}_i, & \text{if } y_{i+1} < \underline{b}_i \end{cases} \qquad (79)$$

then $|\triangle B_{i+1}| = |y_{i+1} - b_i| = |f(y_i)+u_i+w_{i+1} - b_i|$, which is also constant for all $f \in \mathcal{F}_i$.

Let $x_i \triangleq f(b_i)$. Then obviously, x_i is constant for all $f \in \mathcal{F}_i$.

Consequently, under condition C2), for $f \in \mathcal{F}, f(y_{i+1})$ could be any value on the interval

$$[f(b_i) - L|\triangle B_{i+1}|, f(b_i) + L|\triangle B_{i+1}|] = [x_i - L|\triangle B_{i+1}|, x_i + L|\triangle B_{i+1}|].$$

Step 4: Finally, we prove that $\overline{\lim}_{t\to\infty}|y_t| = \infty$.

If this were not true, then both $\bar{b}_\infty \triangleq \lim_{i\to\infty}\bar{b}_i$ and $\underline{b}_\infty \triangleq \lim_{i\to\infty}\underline{b}_i$ would be finite. Introduce a set

$$\mathcal{F}_\infty \triangleq \{f \in \mathcal{F}(L,0)\colon f(y_i) = f_i(y_i) \text{ for } f_i \in \mathcal{F}_i, i \geq 0\}$$

which is well-defined since $y_{i+1} \notin B_i, \forall\, i \geq 0$ implies $y_i \neq y_j$ for $i \neq j$ and since $f_i(y_i)$ is independent of the particular choice of $f_i \in \mathcal{F}_i$.

Now, define a function f_∞ as

$$f_\infty(x) = \begin{cases} \underline{b}_\infty, & x \leq \underline{b}_\infty \\ \text{linear interpolation of } (y_i, f_i(y_i)) \text{ with } f_i \in \mathcal{F}_i, & \underline{b}_\infty < x < \bar{b}_\infty \\ \bar{b}_\infty, & x \geq \bar{b}_\infty. \end{cases}$$

Note that for any $0 \leq i < j < \infty$, by $\mathcal{F}_j \subset \mathcal{F}_i \subset \mathcal{F}(L,0)$, we have for $f_i \in \mathcal{F}_i$ and $f_j \in \mathcal{F}_j$

$$|f_\infty(y_i) - f_\infty(y_j)| = |f_i(y_i) - f_j(y_j)| = |f_j(y_i) - f_j(y_j)| \leq L|y_i - y_j|.$$

Hence it is easy to see that $f_\infty \in \mathcal{F}_\infty$ and thus $\mathcal{F}_\infty \neq \emptyset$.

Also it is obvious that $\mathcal{F}_\infty \subset \mathcal{F}_i, \forall\, i \geq 0$. Consequently, for any $f \in \mathcal{F}_\infty$, by (71), (76), and (78), we have $|\triangle B_1| \geq 1$, and

$$|\triangle B_{i+1}| \geq L|\triangle B_i| - \tfrac{1}{2}|B_i|, \qquad i \geq 1$$

which can be rewritten as

$$|B_{i+1}| \geq L(|B_i| - |B_{i-1}|) + \tfrac{1}{2}|B_i|, \qquad i \geq 1$$

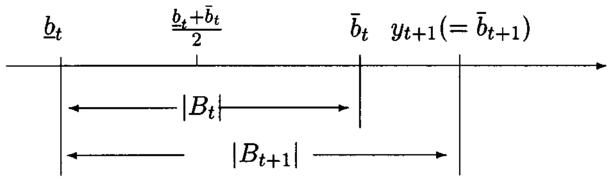

Fig. 4. Illustration of $y_{t+1} \notin B_t$.

by using

$$|\triangle B_{i+1}| = |B_{i+1}| - |B_i|, \qquad i \geq 0$$

with $|B_0| = 0, |B_1| \geq 1$.

Therefore, by Lemma 3.2, we have $\lim_{t\to\infty} |B_t| = \infty$, i.e., $\overline{\lim}_{t\to\infty} |y_t| = \infty$. Hence the conclusion of Theorem 2.1 ii) is true. □

Proof of Theorem 2.2

We divide the proof into five steps.

Step 1: We analyze some properties of the notations (62), (63).

First, it is clear that (see, e.g., Fig. 4)

$$\begin{cases} |B_{t+1}| = |B_t|, & \text{if } y_{t+1} \in B_t \\ |B_{t+1}| = \left|y_{t+1} - \frac{1}{2}(\underline{b}_t + \bar{b}_t)\right| + \frac{1}{2}|B_t|, & \text{if } y_{t+1} \notin B_t. \end{cases}$$

Then since $|y_{t+1} - (1/2)(\underline{b}_t + \bar{b}_t)| > 1/2|B_t| \Longleftrightarrow y_{t+1} \notin B_t$, we have

$$|B_{t+1}| = \max\left\{|y_{t+1} - \tfrac{1}{2}(\underline{b}_t + \bar{b}_t)| + \tfrac{1}{2}|B_t|, |B_t|\right\}. \tag{80}$$

Now we proceed to prove that

$$|y_t - y_{i_t}| \leq \max_{0\leq i\leq t} |\triangle B_i|, \qquad \forall t \geq 1 \tag{81}$$

where i_t is defined in (16). We consider two cases separately.

Case (1): If $y_t \notin B_{t-1}$, then by definitions (15), (16), (62), and (63), we have (see Fig. 4 with $t+1$ replaced by t)

$$|y_t - y_{i_t}| = |B_t| - |B_{t-1}| = |\triangle B_t|.$$

Case (2): If $y_t \in B_{t-1}$, then by (64), we know $y_t \in \triangle B_i$ for some $0 \leq i \leq t-1$. Then by (16) we have

$$|y_t - y_{i_t}| \leq |\triangle B_i| \qquad \text{for the same } i.$$

Combining the two cases above, we see that (81) is true.

Step 2: We proceed to find a recursive inequality on $\{|\triangle B_t|, t \geq 0\}$.

By (1) and (19)–(21), we have

$$y_{t+1} = \begin{cases} f(y_t) - \widehat{f}_t(y_t) + \frac{1}{2}(\underline{b}_t + \bar{b}_t) + w_{t+1}, \\ \qquad \text{if } |y_t - y_{i_t}| > \epsilon \\ f(y_t) - \widehat{f}_t(y_t) + y^*_{t+1} + w_{t+1}, \\ \qquad \text{if } |y_t - y_{i_t}| \leq \epsilon. \end{cases}$$

Hence by (18), we know that if $|y_t - y_{i_t}| > \epsilon$, then

$$y_{t+1} = f(y_t) - f(y_{i_t}) + \tfrac{1}{2}(\underline{b}_t + \bar{b}_t) - w_{i_t+1} + w_{t+1} \tag{82}$$

and if $|y_t - y_{i_t}| \leq \epsilon$, then

$$y_{t+1} = f(y_t) - f(y_{i_t}) + y^*_{t+1} - w_{i_t+1} + w_{t+1}. \tag{83}$$

Now, if for some $t \geq 1, |y_t - y_{i_t}| > \epsilon$, then by (80) and (82),

$$\begin{aligned} &|B_{t+1}| \\ &\quad = \max\left\{|f(y_t) - f(y_{i_t}) - w_{i_t+1} + w_{t+1}| + \tfrac{1}{2}|B_t|, |B_t|\right\}. \end{aligned}$$

Hence by (2) and the definition of $|B_t|$, we have

$$\begin{aligned} |B_t| &\leq |B_{t+1}| \\ &\leq \max\left\{|f(y_t) - f(y_{i_t})| + 2w + \tfrac{1}{2}|B_t|, |B_t|\right\}. \end{aligned}$$

Furthermore, since $f \in \mathcal{F}(L)$ and $L < 3/2 + \sqrt{2}$, by (13) we know that there exist some L_1 satisfying $L < L_1 < 3/2 + \sqrt{2}$ and some $c \geq 0$, such that $f \in \mathcal{F}(L_1, c)$. Hence, by (9) we have

$$\begin{aligned} |B_t| &\leq |B_{t+1}| \\ &\leq \max\left\{L_1 |y_t - y_{i_t}| + c + 2w + \tfrac{1}{2}|B_t|, |B_t|\right\}. \end{aligned} \tag{84}$$

Consequently, by (63), we have

$$0 \leq |\triangle B_{t+1}| \leq \left(L_1 |y_t - y_{i_t}| + c + 2w - \tfrac{1}{2}|B_t|\right)^+.$$

Therefore, by (63) and (81), we have

$$\begin{aligned} 0 &\leq |\triangle B_{t+1}| \\ &\leq \left(L_1 \max_{0\leq i\leq t} |\triangle B_i| + c + 2w - \frac{1}{2}\sum_{i=0}^{t} |\triangle B_i|\right)^+ \end{aligned} \tag{85}$$

which holds for any $t \geq 1$ with $|y_t - y_{i_t}| > \epsilon$.

Step 3: We now prove that for any $s \geq 0$, there exists some $\tau > s$ such that

$$|y_\tau| \leq L_1\epsilon + c + S + 2w \tag{86}$$

where S is defined in (14).

First, there must exist some $\tau' > s$ such that $|y_{\tau'} - y_{i_{\tau'}}| \leq \epsilon$. Otherwise, if

$$|y_t - y_{i_t}| > \epsilon, \qquad \forall t > s \tag{87}$$

then by (85), $\forall t > s$

$$\begin{aligned} 0 &\leq |\triangle B_{t+1}| \\ &\leq \left(L_1 \max_{0\leq i\leq t} |\triangle B_i| + c + 2w - \frac{1}{2}\sum_{i=0}^{t} |\triangle B_i|\right)^+. \end{aligned}$$

Then by Lemma 3.3, $\lim_{t\to\infty} \sum_{i=0}^{t} |\triangle B_i| < \infty$, i.e., $\lim_{t\to\infty} |B_t| < \infty$. Thus by the definition of $|B_t|$, we have $\sup_{t\geq 0} |y_t| < \infty$. Consequently, by Lemma 3.4, we have $\lim_{t\to\infty} |y_t - y_{i_t}| = 0$, which contradicts to (87). Hence, there exists $\tau' > s$ such that $|y_{\tau'} - y_{i_{\tau'}}| \leq \epsilon$. Therefore by (83), we have for $f \in \mathcal{F}(L_1, c)$

$$\begin{aligned} |y_{\tau'+1}| &\leq \left|f(y_{\tau'}) - f(y_{i_{\tau'}}) + y^*_{\tau'+1}\right| + 2w \\ &\leq L_1 \left|y_{\tau'} - y_{i_{\tau'}}\right| + c + S + 2w \\ &\leq L_1\epsilon + c + S + 2w. \end{aligned} \tag{88}$$

From which we arrive at (86) by setting $\tau = \tau' + 1$.

Step 4: We prove the boundedness of the whole sequence $\{y_t, t \geq 0\}$.

Define

$$t_0 \triangleq \inf_{t>0}\{t: |y_t| \leq L_1\epsilon + c + S + 2w\}$$
$$t_n \triangleq \inf_{t>t_{n-1}}\{t: |y_t| \leq L_1\epsilon + c + S + 2w\}, \qquad n \geq 1. \quad (89)$$

Then by (86), we have $t_n < \infty, \forall n \geq 0$.

Let $z_n \triangleq y_{t_n}$. By (89), we have

$$|z_n| \leq L_1\epsilon + c + S + 2w, \qquad \forall n \geq 0.$$

Then by Lemma 3.4 with i_n defined as in (53), we have $\lim_{n\to\infty}|z_n - z_{i_n}| = 0$. Thus there exists some $n_0 \geq 0$ such that for all $n \geq n_0$

$$|z_n - z_{i_n}| \leq \epsilon, \quad \text{i.e.,} \quad \min_{0\leq i\leq n-1}|z_n - z_i| \leq \epsilon.$$

Consequently, for any $n \geq n_0$

$$\left|y_{t_n} - y_{i_{(t_n)}}\right| = \min_{0\leq i\leq t_n-1}|y_{t_n} - y_i| \leq \min_{0\leq i\leq n-1}|y_{t_n} - y_{t_i}| \leq \epsilon$$

where $i_{(t_n)} \triangleq \arg\min_{0\leq i\leq t_n-1}|y_{t_n} - y_i|$.

Hence by (83) we have for any $f \in \mathcal{F}(L_1, c)$

$$|y_{t_n+1}| \leq \left|f(y_{t_n}) - \left(y_{i_{(t_n)}}\right) + y^*_{t_n+1}\right| + 2w \leq L_1\epsilon + c + S + 2w, \qquad \forall n \geq n_0.$$

So by (89)

$$t_{n+1} = t_n + 1, \quad \forall n \geq n_0$$

which implies that

$$|y_t| \leq L_1\epsilon + c + S + 2w, \qquad \forall t > t_{n_0}. \quad (90)$$

Step 5: Finally, we give an upper bound for the asymptotic tracking error.

From (90), by Lemma 3.4 again, we have

$$\lim_{t\to\infty}|y_t - y_{i_t}| = 0. \quad (91)$$

Hence there exists some $T > 0$ such that

$$|y_t - y_{i_t}| \leq \epsilon, \qquad \forall t \geq T.$$

Finally, by (83) and (91), we have for $t \geq T$

$$|y_{t+1} - y^*_{t+1}| \leq |f(y_t) - f(y_{i_t})| + 2w \leq L_1|y_t - y_{i_t}| + c + 2w \to c + 2w, \quad \text{as } t \to \infty$$

which is a tantamount of Theorem 2.2. Hence the proof is completed. □

Proof of Theorem 2.3

First, we rewrite (26) into a recursive form.

Define

$$\triangle_j \triangleq \delta_{j-1} \bigcup \delta_j \bigcup \delta_{j+1}, \qquad j \in Z. \quad (92)$$

Then by (24), we have

$$\triangle(y) = \triangle_j \qquad \text{for } y \in \delta_j,\ j \in Z. \quad (93)$$

Next, for any $j \in Z$, define $N_j(0) \triangleq 0$,

$$N_j(t) \triangleq \sum_{i=0}^{t-1} I_{\triangle_j}(y_i), \qquad t \geq 1. \quad (94)$$

Then for any $j \in Z$, recursively define $\alpha_j(t), t \geq 1$ as

$$\alpha_j(t) \triangleq \begin{cases} \dfrac{\alpha_j(t-1)\cdot N_j(t-1) + (y_t - u_{t-1})I_{\triangle_j}(y_{t-1})}{N_j(t)} & \text{if } N_j(t) > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (95)$$

where $\alpha_j(0) \triangleq 0$.

By simple manipulations, it is easy to see that if for some $t \geq 1, N_j(t) > 0$, then

$$\alpha_j(t) = N_j^{-1}(t)\sum_{i=0}^{t-1}(y_{i+1} - u_i)I_{\triangle_j}(y_i). \quad (96)$$

Now, when $|y_t - y_{i_t}| \leq \epsilon$, by (25), (93), and (94), we have

$$N_j(t) > 0 \qquad \text{for } y_t \in \delta_j,\ j \in Z. \quad (97)$$

Hence by (96), we can rewrite (26) as

$$\overline{f}_t(y_t) = \alpha_j(t), \qquad \text{if } y_t \in \delta_j,\ j \in Z. \quad (98)$$

For $y_t \in \delta_j$ and $y_i \in \triangle_j$, by the definitions of δ_j and $\triangle_j$, we have $|y_t - y_i| \leq 2\epsilon$. Furthermore, since $f \in \mathcal{F}(L)$ with $L < 3/2 + \sqrt{2}$, by (13) we know that there exist some L_1 satisfying $L < L_1 < 3/2 + \sqrt{2}$ and some $c \geq 0$, such that

$$f \in \mathcal{F}(L_1, c).$$

Hence, by the definition (9) we have for $y_t \in \delta_j$ and $y_i \in \triangle_j$

$$|f(y_t) - f(y_i)| \leq L_1|y_t - y_i| + c \leq 2L_1\epsilon + c.$$

Therefore, when $|y_t - y_{i_t}| \leq \epsilon$ and $y_t \in \delta_j, j \in Z$, by (96)–(98) and (1) we have

$$\begin{aligned} &|f(y_t) - \overline{f}_t(y_t)| \\ &\quad = \left|f(y_t) - N_j^{-1}(t)\sum_{i=0}^{t-1}(y_{i+1} - u_i)I_{\triangle_j}(y_i)\right| \\ &\quad \leq \left|N_j^{-1}(t)\sum_{i=0}^{t-1}[f(y_t) - f(y_i)]I_{\triangle_j}(y_i)\right| \\ &\qquad + \left|N_j^{-1}(t)\sum_{i=0}^{t-1} w_{i+1}I_{\triangle_j}(y_i)\right| \\ &\quad \leq 2L_1\epsilon + c + w. \end{aligned} \quad (99)$$

Note that when $|y_t - y_{i_t}| \leq \epsilon$, by (1), (21), and (27) the closed-loop equation is

$$y_{t+1} = f(y_t) - \overline{f}_t(y_t) + y^*_{t+1} + w_{t+1}. \quad (100)$$

Comparing this with (83) and going through the proof of Theorem 2.2, we see that the first four steps of the proof there are also applicable *mutatis mutandis* to the present case. Hence, by the conclusion of Step 4 there, we have for some $M > 0$

$$|y_t| \le M, \qquad t \ge 0. \tag{101}$$

Therefore, by Lemma 3.4 we know that there exists some $t_0 > 0$ such that

$$|y_t - y_{i_t}| \le \epsilon, \qquad \forall\, t \ge t_0. \tag{102}$$

To get the desired result (28), we introduce some notations:

$$N_j \triangleq \lim_{t\to\infty} N_j(t) = \lim_{t\to\infty} \sum_{i=0}^{t-1} I_{\triangle_j}(y_i), \qquad j \in Z$$

$$J_1 \triangleq \{j \in Z\colon 0 < N_j < \infty\}$$

$$J_2 \triangleq \{j \in Z\colon N_j = +\infty\}$$

$$\delta_{J_1} \triangleq \bigcup_{j\in J_1} \delta_j$$

and

$$\delta_{J_2} \triangleq \bigcup_{j\in J_2} \delta_j.$$

By these definitions we know that $y_t \in \delta_{J_1} \bigcup \delta_{J_2}, \forall\, t \ge 0$. Also by (101), we know that $J \triangleq J_1 \bigcup J_2$ is a finite set.

We first analyze the case $y_t \in \delta_{J_1}$.

Denoting $\tilde{f}_t(y_t) \triangleq f(y_t) - \overline{f}_t(y_t)$, by (99) and (102), we have for $t \ge t_0$

$$|\tilde{f}_t(y_t)| \le 2L_1\epsilon + c + w.$$

Hence by

$$\sum_{t=0}^{\infty} I_{\delta_{J_1}}(y_t) = \sum_{j\in J_1} \sum_{t=0}^{\infty} I_{\delta_j}(y_t) < \infty$$

we have

$$\frac{1}{T} \sum_{\substack{0\le t\le T\\ y_t\in\delta_{J_1}}} \tilde{f}_t^2(y_t) \to 0 \qquad \text{as } T\to\infty. \tag{103}$$

Next, let $y_t \in \delta_{J_2}$, then there is some $j \in J_2$ such that $y_t \in \delta_j$.

Similar to (99), when $|y_t - y_{i_t}| \le \epsilon$, we know that for any $\lambda_1 > 1$, there exists some $\lambda_2 > 1$ such that

$$\begin{aligned} &|f(y_t) - \overline{f}_t(y_t)|^2 \\ &\quad\le \lambda_1 \left| N_j^{-1}(t) \sum_{i=0}^{t-1} [f(y_t) - f(y_i)] I_{\triangle_j}(y_i) \right|^2 \\ &\qquad + \lambda_2 \left| N_j^{-1}(t) \sum_{i=0}^{t-1} w_{i+1} I_{\triangle_j}(y_i) \right|^2 \\ &\quad\le \lambda_1 (2L_1\epsilon + c)^2 + \lambda_2 \left| N_j^{-1}(t) \sum_{i=0}^{t-1} w_{i+1} I_{\triangle_j}(y_i) \right|^2. \end{aligned} \tag{104}$$

For any $j \in J_2$, by the law of large numbers for martingales, we have from (22) and $N_j = \infty$ that

$$N_j^{-1}(t) \sum_{i=0}^{t-1} w_{i+1} I_{\triangle_j}(y_i) \to 0, \qquad \text{as } t\to\infty. \tag{105}$$

Then by (102), (104), and (105), we have

$$\begin{aligned} \frac{1}{T} \sum_{\substack{t_0\le t\le T\\ y_t\in\delta_{J_2}}} \tilde{f}_t^2(y_t) &= \frac{1}{T} \sum_{j\in J_2} \sum_{\substack{t_0\le t\le T\\ y_t\in\delta_j}} \tilde{f}_t^2(y_t) \\ &\le \lambda_1 (2L_1\epsilon + c)^2 + \lambda_2 \sum_{j\in J_2} \frac{1}{T} \sum_{\substack{t_0\le t\le T\\ y_t\in\delta_j}} \\ &\quad \cdot \left[N_j^{-1}(t) \sum_{i=0}^{t-1} w_{i+1} I_{\triangle_j}(y_i) \right]^2 \\ &\le \lambda_1 (2L_1\epsilon + c)^2 + o(1). \end{aligned} \tag{106}$$

Consequently, by (100), (102), (103), and (106), we have

$$\begin{aligned} &\frac{1}{T} \sum_{t=0}^{T} (y_{t+1} - y_{t+1}^* - w_{t+1})^2 \\ &\quad = \frac{1}{T} \sum_{t=0}^{t_0-1} (y_{t+1} - y_{t+1}^* - w_{t+1})^2 \\ &\qquad + \frac{1}{T} \sum_{t=t_0}^{T} [f(y_t) - \overline{f}_t(y_t)]^2 \\ &\quad = o(1) + \frac{1}{T} \sum_{\substack{t_0\le t\le T\\ y_t\in\delta_{J_1}}} \tilde{f}_t^2(y_t) + \frac{1}{T} \sum_{\substack{t_0\le t\le T\\ y_t\in\delta_{J_2}}} \tilde{f}_t^2(y_t) \\ &\quad \le o(1) + \lambda_1 (2L_1\epsilon + c)^2. \end{aligned} \tag{107}$$

Finally, since $\lambda_1 > 1$ is arbitrary, by (107) we have

$$\overline{\lim_{T\to\infty}} \frac{1}{T} \sum_{t=0}^{T} (y_{t+1} - y_{t+1}^* - w_{t+1})^2 \le (2L_1\epsilon + c)^2$$

which is the desired conclusion (28). □

Proof of Theorem 2.4

i) Sufficiency: We need only to show that if $f \in \mathcal{G}(L)$ with $L < 4$, then the feedback law defined by (32) is stabilizing.

By (1) we have

$$|f(y_{j_t}) - (y_{j_t+1} - u_{j_t})| = |w_{j_t+1}| \le w.$$

Then by (31), we have

$$\begin{aligned} |f(y_t) - \hat{\hat{f}}_t(y_t)| &= |f(y_t) - (y_{j_t+1} - u_{j_t})| \\ &\le |f(y_t) - f(y_{j_t})| + w. \end{aligned} \tag{108}$$

Now, denote

$$\mathcal{G}(L, c) \triangleq \{f \in \mathcal{F}(L, c)\colon f(x) = f(-x), \forall\, x\}. \tag{109}$$

It easy to verify (cf., Appendix A) that for $\mathcal{G}(L)$ defined by (29)

$$\mathcal{G}(L, c) \subset \mathcal{G}(L), \qquad \forall\, c \ge 0 \tag{110}$$

and that for any $\gamma > 0$,

$$\mathcal{G}(L) \subset \bigcup_{c\in[0,\infty)} \mathcal{G}(L+\gamma, c). \tag{111}$$

Hence for any $f \in \mathcal{G}(L)$ and any $\gamma > 0$, there exists some $c \ge 0$, such that

$$f \in \mathcal{G}(L+\gamma, c). \tag{112}$$

Since $f \in \mathcal{G}(L)$ with $L < 4$, by (112) we know that there exist $L_1 \in (L, 4)$ and $c \geq 0$ such that $f \in \mathcal{G}(L_1, c)$. Hence, by (109), (108), we have

$$|f(y_t) - \hat{\hat{f}}_t(y_t)| \leq L_1 \left||y_t| - |y_{j_t}|\right| + c + w.$$

Furthermore, by (1), (30), and (32), we have

$$\begin{aligned}|y_{t+1} - y^*_{t+1}| &= \left|f(y_t) - \hat{\hat{f}}_t(y_t) + w_{t+1}\right| \\ &\leq L_1 \left||y_t| - |y_{j_t}|\right| + c + 2w \\ &= L_1 \min_{0 \leq i \leq t-1} \left||y_t| - |y_i|\right| + c + 2w.\end{aligned}$$

Then by Lemma 3.7 (setting $h_t = |y_t|$ and $z_t = \mathrm{sgn}(y_t) \cdot y^*_t$), we have

$$\overline{\lim_{t \to \infty}} |y_t - y^*_t| \leq c + 2w$$

which proves the stability part of Theorem 2.4 and also establishes the tracking error bound.

ii) Necessity: We will show that if $L \geq 4$, then for any given feedback control law $\{u_t\}$, there always exists some $f \in \mathcal{G}(L, 0) \subset \mathcal{G}(L)$ such that the corresponding closed-loop system (1) with this $\{u_t\}$ is unstable.

To find such an $f \in \mathcal{G}(L, 0)$ (depending on $\{u_t\}$), our method is to construct a sequence of nonincreasing nonempty sets $\mathcal{G}_t \subset \mathcal{G}(L, 0), t = 0, 1, 2, \cdots$ (depending on $\{u_t\}$) such that $\{|y_i|, 0 \leq i \leq t+1\}$ is strictly increasing for any $f \in \mathcal{G}_t$. The construction techniques are similar to those in the proof of Theorem 2.1 ii), save that Lemma 3.2 there is replaced by Lemma 3.6 and that instead of (67) we start with the following initial sets:

$$\left.\begin{aligned}\mathcal{G}'_0 &\triangleq \{f\colon f(y_0) = (L+1)|y_0|; f \in \mathcal{G}(L, 0)\} \neq \emptyset \\ \mathcal{G}''_0 &\triangleq \{f\colon f(y_0) = -(L+1)|y_0|; f \in \mathcal{G}(L, 0)\} \neq \emptyset\end{aligned}\right\}. \tag{113}$$

The details will be not repeated. □

V. Concluding Remarks

Feedback and uncertainty are two basic concepts in automatic control. To explore both the full capability and the potential limitations of feedback in controlling nonlinear systems with large structural uncertainty is not only of fundamental importance in feedback theory, but also instrumental in understanding how intelligent a control system can be.

In this contribution, a quantitative study on the relationship between these two concepts (defined in the most general way) has been initiated for the benchmark system (1) where the function $f(\cdot)$ is assumed to be completely unknown. By introducing a suitable norm in the space of all $R^1 \to R^1$ mappings, we have established a series concrete results concerning the capability, limitation and performance of feedback. In particular, we have found and demonstrated that the maximum uncertainty that can be dealt with by feedback is a ball with radius $3/2 + \sqrt{2}$ in this normed function space.

There are many problems remain open in this vital field. First, it is desirable to study uncertain nonlinear control systems more complicated than the basic model (1), for example, high-order systems with uncertainties coupled with the input. Second, it would also be of considerable importance to study hybrid control systems consisting of continuous-time nonlinear plants and sampled-data feedback controllers with *prescribed* sampling period. Some progress has been made in this direction recently in [38], but more efforts are still needed. Finally, a more challenging problem is to find a suitable framework within which the issue of establishing a quantitative relationship among *a priori* information, feedback performance and computational complexity can be addressed adequately and rigorously.

Appendix A

Proof of (11) and (12)

For any $f \in \mathcal{F}(L, c)$, by condition C1), we have

$$|f(x) - f(y)| \leq L|x - y| + c, \qquad \forall (x, y) \in \mathbb{R}^2.$$

Then by (4), it is easy to verify that $\|f\| \leq L$, i.e., $f \in \mathcal{F}(L)$. Hence, (11) holds.

To prove (12), let $f \in \mathcal{F}(L)$, we then have $\|f\| \leq L$. Then by (4), for any $\gamma > 0$, there exists some $\alpha_0 > 0$ such that

$$\sup_{(x,y) \in \mathbb{R}^2} \frac{|f(x) - f(y)|}{|x - y| + \alpha_0} \leq L + \gamma.$$

Hence, we have

$$|f(x) - f(y)| \leq (L + \gamma)|x - y| + \alpha_0(L + \gamma), \quad \forall (x, y) \in \mathbb{R}^2.$$

Denote $c_0 = \alpha_0(L + \gamma)$, then $f \in \mathcal{F}(L + \gamma, c_0)$, and (12) holds. □

Appendix B

Proof of Lemmas 3.5–3.7

Proof of Lemma 3.5: The proof idea is similar to that of Lemma 3.1. Suppose that

$$a_n \geq a_{n-1}, \qquad \forall n \geq 1. \tag{114}$$

Then it is obvious that $a_n \geq 1, \forall n \geq 1$ and $x_n \triangleq a_n/a_{n-1} \geq 1$. Hence, dividing both sides of (55) by a_n, we have

$$x_{n+1} \leq L\left(1 - \frac{1}{x_n}\right) + d.$$

Now, if we denote $b \triangleq \underline{\lim}_{n \to \infty} x_n \geq 1$, then we have

$$b \leq L\left(1 - \frac{1}{b}\right) + d.$$

From this, it is easy to see that $b \in (1, \infty)$ provided that $d \in [0, 1)$. Futhermore, the above inequality can be rewritten as

$$L \geq \frac{b^2 - db}{b - 1}. \tag{115}$$

Now, similar to the proof of Lemma 3.1, it can be shown by using $4 > L$ that there exists a positive constant d_0 depending upon L such that whenever $d \in [0, d_0]$

$$\min_{1 < b < \infty} \frac{b^2 - db}{b - 1} > L$$

which obviously contradicts to (115), and hence (114) is not true and the proof of the lemma is completed. □

Proof of Lemma 3.6: Since $\triangle \triangleq L^2 - 4L \geq 0$ for $L \geq 4$, the quadratic equation $x^2 - Lx + L = 0$ has two real roots, which are denoted by λ_1 and λ_2. Then

$$\lambda_1 + \lambda_2 = L \tag{116}$$
$$\lambda_1 \cdot \lambda_2 = L. \tag{117}$$

It is obvious from (116) and (117) that $\lambda_1 > 0$ and $\lambda_2 > 0$. Now we prove that $\lambda_1 > 1$ and $\lambda_2 > 1$. Otherwise, if $\lambda_1 \leq 1$, then $\lambda_2 \geq L$ by (117), which contradicts to (116). Hence $\lambda_1 > 1$. Similarly, $\lambda_2 > 1$.

By (116) and (117) we rewrite (57) as

$$(a_{n+1} - \lambda_1 a_n) \geq \lambda_2(a_n - \lambda_1 a_{n-1}), \qquad n \geq 1.$$

Then

$$(a_{n+1} - \lambda_1 a_n) \geq \lambda_2^n(a_1 - \lambda_1 a_0) \geq \lambda_2^n, \qquad n \geq 1.$$

So by $\lambda_2 > 1$, we have

$$a_{n+1} - \lambda_1 a_n > 0 \quad \text{and} \quad (a_n - \lambda_1 a_{n-1}) \underset{n\to\infty}{\to} \infty. \tag{118}$$

Hence by $\lambda_1 > 1$, (58) holds.

Now by (116), we may assume $\lambda_1 \geq L/2$ (otherwise, $\lambda_2 \geq L/2$, and the arguments are similar). Then by (118) and the fact that $L/2 > L/(L-1)$, for $L \geq 4$, we have

$$\begin{aligned} L(a_n - a_{n-1}) - a_n &= (L-1)a_n - La_{n-1} \\ &= (L-1)\left(a_n - \frac{L}{L-1}\, a_{n-1}\right) \\ &\geq (L-1)(a_n - \lambda_1 a_{n-1}) > 0 \end{aligned}$$

which is precisely (59). □

Proof of Lemma 3.7: First we prove that

$$\overline{\lim_{n\to\infty}}\; h_n < \infty. \tag{119}$$

We adopt the contradiction argument. Suppose $\overline{\lim}_{n\to\infty} h_n = \infty$, then we could choose a strictly increasing subsequence $\{h_{n_j}, j \geq 0\}$ from $\{h_n, n \geq 0\}$ such that $h_{n_0} = h_0$, and $\forall\, j \geq 0$

$$h_{n_{j+1}} > h_{n_j}$$

and

$$h_n \leq h_{n_j} \qquad \text{for } n_j \leq n < n_{j+1}. \tag{120}$$

Let $H_0 \triangleq h_{n_0}, H_{j+1} \triangleq h_{n_{j+1}} - h_{n_j}, j \geq 0$. We now show that

$$\min_{0\leq i\leq n_{j+1}-2} \left|h_{(n_{j+1}-1)} - h_i\right| \leq \max_{0\leq k\leq j} H_k. \tag{121}$$

By (120), we have $h_{(n_{j+1}-1)} \leq h_{n_j}$. So we need only to consider the following two cases:

Case (1): $h_{(n_{j+1}-1)} < h_{n_j}$. In this case, there exists some $0 \leq k_0 \leq j$ such that

$$h_{n_{(k_0-1)}} \leq h_{(n_{j+1}-1)} < h_{n_{(k_0)}} \tag{122}$$

where, $h_{n_{(-1)}} \triangleq 0$. Also we have $n_{j+1} - 1 \neq n_j$, which implies that $n_{j+1} - 2 \geq n_j$. By this and (122), we have

$$\begin{aligned} &\min_{0\leq i\leq n_{j+1}-2} \left|h_{(n_{j+1}-1)} - h_i\right| \\ &\quad\leq \min_{0\leq k\leq j} \left|h_{(n_{j+1}-1)} - h_{n_k}\right| \\ &\quad\leq \left|h_{(n_{j+1}-1)} - h_{n_{k_0}}\right| \leq \left|h_{n_{(k_0-1)}} - h_{n_{k_0}}\right| = H_{k_0}. \end{aligned}$$

Case (2): $h_{(n_{j+1}-1)} = h_{n_j}$. In this case, by $n_{j-1} \leq n_{j+1} - 2$, we have

$$\begin{aligned} \min_{0\leq i\leq n_{j+1}-2} \left|h_{(n_{j+1}-1)} - h_i\right| &\leq \left|h_{(n_{j+1}-1)} - h_{n_{j-1}}\right| \\ &= \left|h_{n_j} - h_{n_{j-1}}\right| = H_j. \end{aligned}$$

Combining Case (1) and Case (2), we see that (121) holds.

Now, by (60) and the boundedness of $\{z_n\}$, we have for some $M > 0$

$$h_{n+1} \leq L \min_{0\leq i\leq n-1} |h_n - h_i| + d + M, \qquad n \geq 1.$$

Then we have

$$h_{n_{j+1}} \leq L \min_{0\leq i\leq n_{j+1}-2} \left|h_{(n_{j+1}-1)} - h_i\right| + d + M.$$

Hence by (121), we have

$$h_{n_{j+1}} \leq L \max_{0\leq k\leq j} H_k + d + M.$$

Therefore, by $h_{n_{j+1}} = \sum_{k=0}^{j} H_k + H_{j+1}$, we have

$$H_{j+1} \leq L \max_{0\leq k\leq j} H_k - \sum_{k=0}^{j} H_k + d + M.$$

From this, by using the same arguments as (47)–(51) in the proof of Lemma 3.3 and by Lemma 3.5, we have for some $J \geq 1$, $H_J < 0$, which contradicts to our definition of H_j. Hence (119) holds.

Consequently, by Lemma 3.4, we have $\min_{0\leq i\leq n-1} |h_n - h_i| \to 0$. Hence (61) follows from (60), and the proof of Lemma 3.7 is completed. □

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