ROBUST STABILITY OF DISCRETE-TIME ADAPTIVE NONLINEAR CONTROL [1]

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Abstract: We consider adaptive control of discrete-time nonlinear systems with unknown parameters and with bounded noise. In the single parameter case, we demonstrate that the necessary and sufficient condition for the existence of a robust feedback stabilizer is that the nonlinear growth rate of the system dynamics is less than 4. This result further confirms the conclusion of (Guo, 1997) where Gaussian noise were considered. In the multiple parameter case, the necessary and sufficient condition turns out to be governed by a polynomial rule, which is identical to the one obtained in (Xie and Guo, 1999) where also Gaussian noise were considered but only the necessity part was proved. To the authors' knowledge, the stabilizing controller constructed in this paper seems the first capable of dealing with any nontrivial noise in the multiple parameter case.

Keywords: Adaptive control, Nonlinear systems, Robust stability, Discrete-time systems, Feedback capability

1. INTRODUCTION

On nonlinear adaptive control, much less results are available in the literature for discrete-time systems, compared with continuous-time systems. The difficulty involved with adaptive control of discrete-time nonlinear systems was clearly demonstrated by the negative conclusion drawn in (Guo, 1997), which states that it is impossible in general to stabilize a discrete-time nonlinear system with even only one unknown parameter if the nonlinear growth rate is too high. In contrast, for a continuous-time counter-part, no matter how high the nonlinear growth rate is, it can always be stabilized by, say, a nonlinear damping controller with a higher order.

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The benchmark model considered by (Guo, 1997) is as follows:

$$y_{t+1} = \theta y_t^b + u_t + w_{t+1}, \quad t = 0, 1, \ldots \tag{1}$$

where, u_t, y_t and w_t are the system input, output and noise respectively, θ is an unknown parameter, and the exponent $b \geq 1$ is a known real number and is regarded as the nonlinear growth rate of the system.

For the system (1), under the assumption that both the unknown parameter θ and the noise $\{w_t\}$ are Gaussian distributed, (Guo, 1997) proved that if the nonlinear growth rate $b \geq 4$, then for any causal feedback control, there always exists a set with positive probability, on which the closed-loop dynamics is unstable. On the other hand, if $b < 4$, it was shown in (Guo, 1997) that the standard least-square-based adaptive control scheme can ensure the closed-loop stability almost surely.

Later on, (Xie and Guo, 1999) extended the negative conclusion of (Guo, 1997) to systems with multiple unknown parameters:

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_p y_t^{b_p} + u_t + w_{t+1} \quad (2)$$

and proved a polynomial rule: (4) is not almost surely stabilizable if $P(x) < 0$ for some $x \in [1, b_1]$, where

$$P(x) = x^{p+1} - b_1 x^p + (b_1 - b_2)x^{p-1} + \cdots + b_p \quad (3)$$

which implies that generally linear growth condition is indispensable for almost sure stability if no constraint is exerted on the number of unknown parameters. This negative polynomial rule was further extended in (Xie and Guo, 2000*a*) to the case where the uncertain parameters are known *a priori* to lie in a bounded region and the systems are allowed to have more general structures:

$$y_{t+1} = \theta^T f(y_t, y_{t-1}, \ldots, y_{t-p+1}) + u_t + w_{t+1} \quad (4)$$

All these results mentioned above assume Gaussian distributed noise. It would be interesting to ask what happens if the noise are bounded. Are there still negative conclusions that prevent the existence of a stabilizing feedback controller for any nonlinear growth rate? If yes, do they have the same constraints on the nonlinear growth rate?

We still take the model (1) as the starting point to answer these questions. Here in this paper, instead, we assume bounded noise. One may suspect that the boundedness assumption on the noise w_t would be helpful for designing feedback stabilizers, which would at least result in a looser requirement on the nonlinear growth rate b. In fact, we will demonstrate the contrary. We will show that $b < 4$ is still necessary for the existence of a feedback stabilizer, even if the noise are assumed to be uniformly bounded and with the bound known *a priori*. However, the boundedness assumption on the noise will indeed be helpful in designing much simpler feedback stabilizers when $b < 4$.

In the multiple parameter case, interestingly, we not only can show the necessity but also the sufficiency of the polynomial rule (3) when bounded noise are considered. Moreover, to the authors' knowledge, the stabilizing controller constructed in this paper seems the first capable of dealing with any nontrivial noise in the multiple parameter case. This certainly raises the question whether the polynomial rule (3) is also the sufficient condition for the Gaussian noise case. This calls for further study.

Other related works include some papers considering noise-free models (see e.g. (Guo and Wei, 1996; Kanellakopoulos, 1994; Zhao and Kanellakopoulos, 2002)). But there is a fundamental drawback with such models. The trick lies in that without noise, the parameters are completely solvable with linear equations. For example, without w_1 in (1), θ can be completely determined by y_1, y_0 and u_0 with the equation

$$y_1 = \theta y_0^b + u_0.$$

Of course this kind of equation-solving methods are not useful in practice due to that they are not robust to the noise. While all the authors realized this and thereby came up with some other (mostly recursive) types of parameter estimation algorithms, it is always impossible to justify the robustness of those algorithms thus obtained without explicitly considering noise. A good example is the weighted-least-square-based adaptive controller constructed in (Kanellakopoulos, 1994), which was shown capable of stabilizing (1) for any nonlinear growth rate $b \geq 1$, but with all $w_t = 0$. Interestingly, one implication of the results in this paper is just that such a controller cannot be robust to bounded noise at least for the case $b \geq 4$.

2. MAIN RESULTS

2.1 One parameter case

Consider the system (1) with the following assumptions.

A1) At the time $t = 0$, the *a priori* knowledge about the unknown parameter θ is that it can be any value on some interval $[\underline{\theta}, \overline{\theta}] \subset \mathbb{R}^1$ with $\overline{\theta} - \underline{\theta} > 0$.

A2) The noise are assumed to be uniformly bounded with the bound $w > 0$, i.e.,

$$\sup_{t \geq 1} |w_t| \leq w. \quad (5)$$

We are interested in designing a feedback control law which robustly stabilizes the system (1) with respect to any possible θ and $\{w_t\}$ under the assumptions A1)-A2).

First, we restate the definition of a feedback control law, which has appeared in (Xie and Guo, 2000*b*).

Definition 2.1. A sequence $\{u_t\}$ is called a feedback control law if at any time $t \geq 0$, u_t is a (causal) function of all the observations up to the time t: $\{y_i, i \leq t\}$, i.e.,

$$u_t = h_t(y_0, \cdots, y_t) \quad (6)$$

where $h_t(\cdot) : \mathbb{R}^{t+1} \to \mathbb{R}^1$ can be any (nonlinear) mapping.

Although there is no unified definition of adaptive control, it is generally thought of as a combination of two parts: online parameter estimation plus controller design with updated parameter estimates. Anyway, it must be causal. That is, whatever the adaptive control law designed, it is one feedback control law in Definition 2.1.

Definition 2.2. The system (1) under the assumptions A1)-A2) is said to be robust feedback stabilizable, if there exists a feedback control law $\{u_t\}$ such that for any $y_0 \in \mathbb{R}^1$ and any θ, $\{w_t\}$ satisfying A1)-A2), the outputs of the closed-loop system are uniformly bounded as follows:

$$\sup_{t\geq 0} |y_t| < \infty. \tag{7}$$

Theorem 2.1. The system (1) under the assumptions A1)-A2) is robust feedback stabilizable if and only if $b < 4$.

Next, consider a more general model:

$$y_{t+1} = \theta f(y_t) + u_t + w_{t+1} \tag{8}$$

where $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a known nonlinear mapping. We consider the following assumptions on $f(\cdot)$.

A3) There exist $a_1 > 0$, $b > 0$ and $M_1 > 0$ such that

$$|f(x)| \leq a_1|x|^b, \quad \text{for } |x| \geq M_1; \tag{9}$$

$$\text{and} \quad \sup_{|x|\leq M_1} |f(x)| < \infty. \tag{10}$$

A4) There exist $a_2 > 0$, $b > 0$ and $M_2 > 0$ such that

$$|f(x)| \geq a_2|x|^b, \quad \text{for } |x| \geq M_2. \tag{11}$$

Intuitively, the assumption A3) exerts an upper bound on the growth rate of the system dynamics; and conversely, the assumption A4) exerts a lower bound on the growth rate of the system dynamics. For either of them, we have one corresponding conclusion as follows.

Theorem 2.2. The system (8) under the assumptions A1)-A3) is robust feedback stabilizable if $b < 4$.

Theorem 2.3. The system (8) under the assumptions A1)-A2) and A4) is not robust feedback stabilizable if $b \geq 4$.

The combination of Theorems 2.2 and 2.3 immediately leads to the following conclusion.

Corollary 2.1. The system (8) under the assumptions A1)-A4) is robust feedback stabilizable if and only if $b < 4$.

2.2 Multiple parameter case

Consider the system

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_p y_t^{b_p} + u_t + w_{t+1} \tag{12}$$

where $p \geq 2$, and the exponents are real numbers and are arranged in decreasing order: $b_1 > b_2 > \cdots > b_p > 0$ with $b_1 > 1$, and the unknown parameters and the noise are assumed to satisfy:

A5) At the time $t = 0$, the *a priori* knowledge about the unknown parameter θ_i is that it can be any value on some interval $[\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}^1$ with $\overline{\theta}_i - \underline{\theta}_i > 0$, for any $i = 1, 2, \ldots, p$.

A6) The noise are assumed to be uniformly bounded with the bound $w > 0$, i.e.,

$$\sup_{t\geq 1} |w_t| \leq w. \tag{13}$$

Theorem 2.4. The system (12) under the assumptions A5)-A6) is robust feedback stabilizable if and only if

$$P(x) > 0 \text{ for any } x \in [1, b_1] \tag{14}$$

where

$$P(x) = x^{p+1} - b_1 x^p + (b_1 - b_2)x^{p-1} + \cdots + (b_{p-1} - b_p)x + b_p$$

Remark 2.1. For $p = 1$, $P(x) = x^2 - b_1 x + b_1$. Then the condition (14) is equivalent to $b_1 < 4$.

At last, we remark that extensions to systems with more complex structures can be done similarly as in the one parameter case.

3. PROOF OF THE THEOREMS

The proofs of Theorems 2.1-2.3 can be found in (Li and Xie, 2004).

Before presenting the proof of Theorem 2.4, we first introduce two lemmas.

Let

$$z = \left(p!\frac{p}{p-1}\right)^{\frac{1}{\min_{1\leq k\leq p-1}(b_k - b_{k+1})}}.$$

Lemma 3.1. Let $a_i \in \mathbb{R}^1, i = 1, 2, \ldots, p$ satisfy $|a_i| > z|a_{i+1}|, i = 1, 2, \cdots, p-1$ and $|a_p| \geq 1$. Let

$$D = \begin{vmatrix} a_1^{b_1} & a_1^{b_2} & \cdots & a_1^{b_p} \\ a_2^{b_1} & a_2^{b_2} & \cdots & a_2^{b_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_p^{b_1} & a_p^{b_2} & \cdots & a_p^{b_p} \end{vmatrix}$$

then we have

$$\frac{1}{p}\prod_{s=1}^{p} |a_s|^{b_s} < |D| < 2\prod_{s=1}^{p} |a_s|^{b_s}$$

Proof: Obviously, D is a summation of terms of the form

$$(-1)^r \prod_{s=1}^{p} a_s^{b_{j_s}} \tag{15}$$

where $r = 0$ or 1, $(j_1, j_2, \dots, j_p) = \pi(1, 2, \dots, p)$, and $\pi\underline{X}$ denotes a permutation of vector $\underline{X}$. Taking logarithm on the absolute value of (15), we have: $\log|\prod_{s=1}^{p} a_s^{b_{j_s}}| = \sum_{s=1}^{p} b_{j_s} \log|a_s|$. Since

$$b_1 > b_2 > \cdots > b_p$$
$$\log|a_1| > \log|a_2| > \cdots > \log|a_p|$$

by the inequality in (Mitrinovic, 1970, p.341), we have

$$\sum_{s=1}^{p} b_{j_s} \log|a_s| \leq \sum_{s=1}^{p} b_s \log|a_s|$$

which means $|a_1^{b_1} a_2^{b_2} a_3^{b_3} \cdots a_p^{b_p}|$ is the maximum term. For any other $\prod_{s=1}^{p} |a_s|^{b_{j_s}}$ with $(b_{j_1}, \dots, b_{j_p}) \neq (b_1, \dots, b_p)$, let j_m be the first one different from m in $\{j_1, j_2, \dots, j_p\}$, $1 \leq m \leq p-1$, i.e., $j_s = s$ for $s < m$, and $j_m = n > m$, then $\left|\prod_{s=1}^{p} a_s^{b_s}\right|$ is no larger than

$$\left|a_1^{b_1} \cdots a_{m-1}^{b_{m-1}} a_m^{b_n} a_{m+1}^{b_m} a_{m+2}^{b_{m+1}} \cdots a_n^{b_{n-1}} a_{n+1}^{b_{n+1}} \cdots a_p^{b_p}\right|$$

for which, we have the following uniform bound:

$$\begin{aligned}
&\frac{\left|\prod_{s=1}^{p} a_s^{b_s}\right|}{\left|a_1^{b_1} \cdots a_{m-1}^{b_{m-1}} a_m^{b_n} a_{m+1}^{b_m} a_{m+2}^{b_{m+1}} \cdots a_n^{b_{n-1}} a_{n+1}^{b_{n+1}} \cdots a_p^{b_p}\right|} \\
&\geq \left|a_m^{b_m - b_n} a_{m+1}^{b_{m+1} - b_m} \cdots a_n^{b_n - b_{n-1}}\right| \\
&= \left|a_m^{b_m - b_{m+1} + \cdots + b_{n-1} - b_n} a_{m+1}^{b_{m+1} - b_m} \cdots a_n^{b_n - b_{n-1}}\right| \\
&\geq \beta^{\delta_m} \beta^{2\delta_{m+1}} \cdots \beta^{(n-m)\delta_{n-1}} \\
&\geq \beta^{(n-m)(n-m+1)\delta/2} \\
&\geq \beta^{\delta} \\
&= p! \frac{p}{p-1}
\end{aligned}$$

where, $\delta_m := b_m - b_{m+1}$, $m = 1, \dots, p-1$ and $\delta := \min_{1 \leq k \leq p-1} \delta_m$. So we have

$$\begin{aligned}
|D| &\geq \prod_{s=1}^{p} |a_s|^{b_s} \left(1 - \sum_{\substack{(j_1,\dots,j_p)=\pi(1,\dots,p) \\ (j_1,\dots,j_p)\neq(1,\dots,p)}} \frac{|\prod_{s=1}^{p} a_s^{b_{j_s}}|}{|\prod_{s=1}^{p} a_s^{b_s}|}\right) \\
&> \prod_{s=1}^{p} |a_s|^{b_s} \left(1 - (p!-1)\frac{p-1}{pp!}\right) \\
&> \frac{1}{p} \prod_{s=1}^{p} |a_s|^{b_s}
\end{aligned}$$

and similarly,

$$\begin{aligned}
|D| &< \prod_{s=1}^{p} |a_s|^{b_s} \left(1 + (p!-1)\frac{p-1}{pp!}\right) \\
&< 2 \prod_{s=1}^{p} |a_s|^{b_s}
\end{aligned}$$

Lemma 3.2. Under the conditions of Lemma 3.1, let $D_{k,l}$ be the kl-th minor of D, i.e.,

$$D_{k,l} = \begin{vmatrix}
a_1^{b_1} & a_1^{b_2} & \cdots & a_1^{b_{l-1}} & a_1^{b_{l+1}} & \cdots & a_1^{b_p} \\
a_2^{b_1} & a_2^{b_2} & \cdots & a_2^{b_{l-1}} & a_2^{b_{l+1}} & \cdots & a_2^{b_p} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{k-1}^{b_1} & a_{k-1}^{b_2} & \cdots & a_{k-1}^{b_{l-1}} & a_{k-1}^{b_{l+1}} & \cdots & a_{k-1}^{b_p} \\
a_{k+1}^{b_1} & a_{k+1}^{b_2} & \cdots & a_{k+1}^{b_{l-1}} & a_{k+1}^{b_{l+1}} & \cdots & a_{k+1}^{b_p} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_p^{b_1} & a_p^{b_2} & \cdots & a_p^{b_{l-1}} & a_p^{b_{l+1}} & \cdots & a_p^{b_p}
\end{vmatrix}$$

we have

$$\begin{aligned}
\frac{1}{p} \prod_{s=1}^{l-1} |a_s|^{b_s} \prod_{s=l}^{p-1} |a_s|^{b_{s+1}} &< \sum_{k=1}^{p} |D_{k,l}| \\
&< 2 \prod_{s=1}^{l-1} |a_s|^{b_s} \prod_{s=l}^{p-1} |a_s|^{b_{s+1}}
\end{aligned} \tag{16}$$

Proof: The terms of $D_{k,l}, k = 1, 2, \dots, p$ have the form

$$(-1)^r \prod_{s=1}^{p} a_s^{b_{j_s}}$$

where $r = 0$ or 1, $(j_1, j_2, \dots, j_p) = \pi(1, \dots, l-1, l+1, \dots, p+1)$. Since

$$b_1 > \cdots > b_{l-1} > b_{l+1} > \cdots > b_p > b_{p+1} = 0$$
$$\log|a_1| > \log|a_2| > \cdots > \log|a_p|$$

Hence, the term in $\sum_{k=1}^{p} |D_{k,l}|$ with the maximum absolute value is $\prod_{s=1}^{l-1} |a_s|^{b_s} \prod_{s=l}^{p-1} |a_s|^{b_{s+1}}$. There are $(p!-1)$ other terms $\prod_{s=1}^{p} |a_s|^{b_{j_s}}$ with $(b_{j_1}, \dots, b_{j_p}) \neq (b_1, \dots, b_{l-1}, b_{l+1}, \dots, b_{p+1})$ and each of them is of an absolute value less than

$$\frac{p-1}{p!p} \prod_{s=1}^{l-1} |a_s|^{b_s} \prod_{s=l}^{p-1} |a_s|^{b_{s+1}}$$

which can be similarly proved as in Lemma 3.1. So (16) follows immediately.

Proof of Theorem 2.4.

For any $t \geq 1$, let

$$i_1(t) := \operatorname*{argmax}_{0 \leq i \leq t-1} |y_i|. \tag{17}$$

$$i_j(t) := \operatorname*{argmax}_{\substack{0 \leq i \leq t-1 \\ z|y_i| < |y_{i_{j-1}(t)}|}} |y_i|. \qquad 2 \leq j \leq p \tag{18}$$

and

$$|y_{i_p(t)}| \geq 1 \tag{19}$$

Let $u_0 = u_1 = \cdots = u_{p-2} = 0$. Starting with $t = p$, if $i_j(p), 1 \leq j \leq p$ as defined in (17)-(19) can not be found, then let $u_{t-1} = 0$, $t = p, p+1, \dots$ until $i_j(t), 1 \leq j \leq p$ can be found. If $i_j(t)$ can never be found for any t, then it is easy to show that $\sup_{t\geq 0}|y_t| < \infty$. We can prove this by contradiction. In fact, if $\sup_{t\geq 0}|y_t| = \infty$, then it is easy to find $k_i, i = 1, 2, \ldots, p$ such that $|y_{k_1}| \geq 1$ and $z|y_{k_{i-1}}| < |y_{k_i}|, i = 2, \ldots, p$. Obviously, for $t = k_p + 1$, $i_j(t)$ in (17)-(19) are well defined. Moreover, it is obvious that $i_j(t)$ are well defined for all $t > k_p + 1$.

So we only need to consider the case where starting from some t_0, $i_j(t)$ in (17)-(19) are all well defined. Then for any $t \geq t_0$, we have

$$
\begin{aligned}
y_{i_1(t)+1} &= \theta_1 y_{i_1(t)}^{b_1} + \cdots + \theta_p y_{i_1(t)}^{b_p} + u_{i_1(t)} + w_{i_1(t)+1}\\
y_{i_2(t)+1} &= \theta_1 y_{i_2(t)}^{b_1} + \cdots + \theta_p y_{i_2(t)}^{b_p} + u_{i_2(t)} + w_{i_2(t)+1}\\
&\vdots\\
y_{i_p(t)+1} &= \theta_1 y_{i_p(t)}^{b_1} + \cdots + \theta_p y_{i_p(t)}^{b_p} + u_{i_p(t)} + w_{i_p(t)+1}
\end{aligned}
$$

That is

$$
\begin{pmatrix} y_{i_1(t)}^{b_1} & y_{i_1(t)}^{b_2} & \cdots & y_{i_1(t)}^{b_p}\\ y_{i_2(t)}^{b_1} & y_{i_2(t)}^{b_2} & \cdots & y_{i_2(t)}^{b_p}\\ \vdots & \vdots & \ddots & \vdots\\ y_{i_p(t)}^{b_1} & y_{i_p(t)}^{b_2} & \cdots & y_{i_p(t)}^{b_p} \end{pmatrix} \begin{pmatrix} \theta_1\\ \theta_2\\ \vdots\\ \theta_p \end{pmatrix} = \begin{pmatrix} y_{i_1(t)+1} - u_{i_1(t)} - w_{i_1(t)+1}\\ y_{i_2(t)+1} - u_{i_2(t)} - w_{i_2(t)+1}\\ \vdots\\ y_{i_p(t)+1} - u_{i_p(t)} - w_{i_p(t)+1} \end{pmatrix} \tag{20}
$$

Let

$$
D(t) = \begin{vmatrix} y_{i_1(t)}^{b_1} & y_{i_1(t)}^{b_2} & \cdots & y_{i_1(t)}^{b_p}\\ y_{i_2(t)}^{b_1} & y_{i_2(t)}^{b_2} & \cdots & y_{i_2(t)}^{b_p}\\ \vdots & \vdots & \ddots & \vdots\\ y_{i_p(t)}^{b_1} & y_{i_p(t)}^{b_2} & \cdots & y_{i_p(t)}^{b_p} \end{vmatrix}
$$

and let $D_l(t)$ denote $D(t)$ with the l-th column replaced by the R.H.S of (20).

By (17)-(19) and Lemma 3.1, we have

$$
|D(t)| > \frac{1}{p}\prod_{s=1}^{p}|y_{i_s(t)}|^{b_s} > 0. \tag{21}
$$

Hence by the Cramer principle,

$$
\theta_l = \frac{D_l(t)}{D(t)}
$$

At the time t, let the parameter estimate be

$$
\hat{\theta}_l(t) \triangleq \frac{\hat{D}_l(t)}{D(t)},
$$

with

$$
\hat{D}_l(t) = \begin{vmatrix} y_{i_1(t)}^{b_1} & \cdots & y_{i_1(t)}^{b_{l-1}} & y_{i_1(t)+1} - u_{i_1(t)} & y_{i_1(t)}^{b_{l+1}} & \cdots & y_{i_1(t)}^{b_p}\\ y_{i_2(t)}^{b_1} & \cdots & y_{i_2(t)}^{b_{l-1}} & y_{i_2(t)+1} - u_{i_2(t)} & y_{i_2(t)}^{b_{l+1}} & \cdots & y_{i_2(t)}^{b_p}\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ y_{i_p(t)}^{b_1} & \cdots & y_{i_p(t)}^{b_{l-1}} & y_{i_p(t)+1} - u_{i_p(t)} & y_{i_p(t)}^{b_{l+1}} & \cdots & y_{i_p(t)}^{b_p} \end{vmatrix}
$$

where all the data are available at time t. Let $\tilde{\theta}_l(t) = \theta - \hat{\theta}_l(t)$. Hence, $\tilde{\theta}_l(t) = \dfrac{\tilde{D}_l(t)}{D(t)}$ with

$$
\tilde{D}_l(t) = \begin{vmatrix} y_{i_1(t)}^{b_1} & \cdots & y_{i_1(t)}^{b_{l-1}} & -w_{i_1(t)+1} & y_{i_1(t)}^{b_{l+1}} & \cdots & y_{i_1(t)}^{b_p}\\ y_{i_2(t)}^{b_1} & \cdots & y_{i_2(t)}^{b_{l-1}} & -w_{i_2(t)+1} & y_{i_2(t)}^{b_{l+1}} & \cdots & y_{i_2(t)}^{b_p}\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ y_{i_p(t)}^{b_1} & \cdots & y_{i_p(t)}^{b_{l-1}} & -w_{i_p(t)+1} & y_{i_p(t)}^{b_{l+1}} & \cdots & y_{i_p(t)}^{b_p} \end{vmatrix}
$$

Let $D_{k,l}(t)$ be the kl-th minor of $D(t)$, i.e. by taking out the k-th row and the l-th column of $D(t)$:

$$
D_{k,l}(t) = \begin{vmatrix} y_{i_1(t)}^{b_1} & y_{i_1(t)}^{b_2} & \cdots & y_{i_1(t)}^{b_{l-1}} & y_{i_1(t)}^{b_{l+1}} & \cdots & y_{i_1(t)}^{b_p}\\ y_{i_2(t)}^{b_1} & y_{i_2(t)}^{b_2} & \cdots & y_{i_2(t)}^{b_{l-1}} & y_{i_2(t)}^{b_{l+1}} & \cdots & y_{i_2(t)}^{b_p}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ y_{i_{k-1}(t)}^{b_1} & y_{i_{k-1}(t)}^{b_2} & \cdots & y_{i_{k-1}(t)}^{b_{l-1}} & y_{i_{k-1}(t)}^{b_{l+1}} & \cdots & y_{i_{k-1}(t)}^{b_p}\\ y_{i_{k+1}(t)}^{b_1} & y_{i_{k+1}(t)}^{b_2} & \cdots & y_{i_{k+1}(t)}^{b_{l-1}} & y_{i_{k+1}(t)}^{b_{l+1}} & \cdots & y_{i_{k+1}(t)}^{b_p}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ y_{i_p(t)}^{b_1} & y_{i_p(t)}^{b_2} & \cdots & y_{i_p(t)}^{b_{l-1}} & y_{i_p(t)}^{b_{l+1}} & \cdots & y_{i_p(t)}^{b_p} \end{vmatrix}
$$

Hence, the estimation error is

$$
\tilde{\theta}_l(t) = \sum_{k=1}^{p}(-1)^{k+l+1} w_{i_k(t)+1}\frac{D_{k,l}(t)}{D(t)} \tag{22}
$$

By (17)-(19), (21), (22) and Lemma 3.2, we have

$$
\begin{aligned}
|\tilde{\theta}_l(t) y_t^{b_l}| &\leq \frac{\sum_{k=1}^{p}|D_{k,l}(t)|}{|D(t)|} w|y_t|^{b_l}\\
&< \frac{2p\prod_{s=1}^{l-1}|y_{i_s(t)}|^{b_s}\cdot\prod_{s=l}^{p-1}|y_{i_s(t)}|^{b_{s+1}}}{\frac{1}{p}\prod_{s=1}^{p}|y_{i_s(t)}|^{b_s}} w|y_t|^{b_l}\\
&= 2p^2 w\left|\frac{y_t}{y_{i_l(t)}}\right|^{b_l}\prod_{s=l}^{p-1}\left|\frac{y_{i_s(t)}}{y_{i_{s+1}(t)}}\right|^{b_{s+1}}
\end{aligned} \tag{23}
$$

Now we define

$$
u_t = -\sum_{l=1}^{p}\hat{\theta}_l(t)\cdot y_t^{b_l} \quad \text{for any } t \geq t_0 \tag{24}
$$

So the closed-loop dynamics is

$$
y_{t+1} = \sum_{l=1}^{p}\tilde{\theta}_l(t)\cdot y_t^{b_l} + w_{t+1} \tag{25}
$$

We use a contradiction argument to prove that $\sup_{t\geq 0}|y_t| < \infty$. Suppose there exist some $y_0 \in \mathbb{R}^1$, $\{\theta_l, l = 1, 2, \cdots, p\}$ and a sequence of $\{w_t\}$, such that for the control defined in (24)

$$
\sup_{t\geq 0}|y_t| = \infty.
$$

From this sequence $\{|y_t|, t \geq t_0\}$, we can pick out a monotonously increasing subsequence $\{|y_{t_k}|, k \geq 1\}$ with

$$|y_{t_1}| > 3p^3 z^{b_1} w \tag{26}$$

$$t_{k+1} = \inf\{t > t_k : |y_t| > z|y_{t_k}|\} \tag{27}$$

For any $k \geq p+1$, let $m = t_{k+1} - 1$, and it is easy to check that

$$|y_m| \leq z|y_{t_k}| \tag{28}$$

$$|y_{t_{k-1}}| \leq |y_{i_1(m)}| \leq z|y_{t_k}| \tag{29}$$

$$|y_{t_{k-j}}| \leq |y_{i_j(m)}| \quad \text{for any } j = 1, 2, \ldots, p \tag{30}$$

In fact, (28) is obvious, and (29) follows by $t_{k-1} \leq t_k - 1 \leq t_{k+1} - 2 = m - 1$, and (30) can be proved by induction: By (29),

$$z|y_{t_{k-2}}| < |y_{t_{k-1}}| \leq |y_{i_1(m)}| \Rightarrow |y_{t_{k-2}}| \leq |y_{i_2(m)}|$$

and this can be continued for $j = 3, 4, \ldots, p$.

Hence by (23), (28)-(30), for any $k \geq p+1$, we have

$$\begin{aligned}
|y_{t_{k+1}}| &\leq \sum_{l=1}^{p} |\tilde{\theta}_l(m)||y_m|^{b_l} + w \\
&\leq 2p^2 w \sum_{l=1}^{p} \Big|\frac{y_m}{y_{i_l(m)}}\Big|^{b_l} \prod_{s=l}^{p-1} \Big|\frac{y_{i_s(m)}}{y_{i_{s+1}(m)}}\Big|^{b_{s+1}} + w \\
&= 2p^2 w \sum_{l=1}^{p} |y_m|^{b_l} \prod_{s=l}^{p-1} \frac{1}{|y_{i_s(m)}|^{b_s - b_{s+1}}} \frac{1}{|y_{i_p(m)}|^{b_p}} + w \\
&\leq 2p^2 w z^{b_1} \sum_{l=1}^{p} |y_{t_k}|^{b_l} \prod_{s=l}^{p-1} \frac{1}{|y_{t_{k-s}}|^{b_s - b_{s+1}}} \frac{1}{|y_{t_{k-p}}|^{b_p}} + w \\
&= 2p^2 w z^{b_1} \sum_{l=1}^{p} \Big|\frac{y_{t_k}}{y_{t_{k-l}}}\Big|^{b_l} \prod_{s=l}^{p-1} \Big|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\Big|^{b_{s+1}} + w \\
&\leq 2z^{b_1} p^3 w \Big|\frac{y_{t_k}}{y_{t_{k-1}}}\Big|^{b_1} \Big|\frac{y_{t_{k-1}}}{y_{t_{k-2}}}\Big|^{b_2} \cdots \Big|\frac{y_{t_{k-p+1}}}{y_{t_{k-p}}}\Big|^{b_p} + w
\end{aligned} \tag{31}$$

where the last inequality follows from the monotonicity of the terms

$$\begin{aligned}
&\Big|\frac{y_{t_k}}{y_{t_{k-l-1}}}\Big|^{b_{l+1}} \prod_{s=l+1}^{p-1} \Big|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\Big|^{b_s+1} \\
&= \Big|\frac{y_{t_{k-l}}}{y_{t_k}}\Big|^{b_l - b_{l+1}} \Big|\frac{y_{t_k}}{y_{t_{k-l}}}\Big|^{b_l} \prod_{s=l}^{p-1} \Big|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\Big|^{b_{s+1}} \\
&< \Big(\frac{1}{z}\Big)^{b_l - b_{l+1}} \Big|\frac{y_{t_k}}{y_{t_{k-l}}}\Big|^{b_l} \prod_{s=l}^{p-1} \Big|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\Big|^{b_{s+1}} \\
&< \Big|\frac{y_{t_k}}{y_{t_{k-l}}}\Big|^{b_l} \prod_{s=l}^{p-1} \Big|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\Big|^{b_{s+1}} \quad \forall\, l = 1, \ldots, p-1
\end{aligned}$$

Since $z^{b_1} p^3 \Big|\frac{y_{t_k}}{y_{t_{k-1}}}\Big|^{b_1} \Big|\frac{y_{t_{k-1}}}{y_{t_{k-2}}}\Big|^{b_2} \cdots \Big|\frac{y_{t_{k-p+1}}}{y_{t_{k-p}}}\Big|^{b_p} > 1$, by (31) we have

$$|y_{t_{k+1}}| \leq 3z^{b_1} p^3 w \Big|\frac{y_{t_k}}{y_{t_{k-1}}}\Big|^{b_1} \Big|\frac{y_{t_{k-1}}}{y_{t_{k-2}}}\Big|^{b_2} \cdots \Big|\frac{y_{t_{k-p+1}}}{y_{t_{k-p}}}\Big|^{b_p} \tag{32}$$

Let $a_k = \ln|y_{t_k}| - \ln 3z^{b_1}p^3 w > 0$. By (26)-(27), $\{a_k\}$ is monotone increasing with $a_1 > 0$ and by (32), we have

$$\begin{aligned}
a_{k+1} \leq\ & b_1(a_k - a_{k-1}) + b_2(a_{k-1} - a_{k-2}) \\
&+ \cdots + b_p(a_{k-p+1} - a_{k-p})
\end{aligned}$$

Then

$$\begin{aligned}
\frac{a_{k+1}}{a_k} \leq\ & b_1\left(1 - \frac{1}{\frac{a_k}{a_{k-1}}}\right) + b_2\left(\frac{1}{\frac{a_k}{a_{k-1}}} - \frac{1}{\frac{a_k}{a_{k-1}}\frac{a_{k-1}}{a_{k-2}}}\right) \\
&+ \cdots + b_p\left(\frac{1}{\prod_{s=k-p+2}^{k} \frac{a_s}{a_{s-1}}} - \frac{1}{\prod_{s=k-p+1}^{k} \frac{a_s}{a_{s-1}}}\right)
\end{aligned}$$

Let $x_k = \frac{a_k}{a_{k-1}}$. Obviously, $x_k > 1$ and we have

$$\begin{aligned}
x_{k+1} \leq\ & b_1 - (b_1 - b_2)\frac{1}{x_k} - (b_2 - b_3)\frac{1}{x_k x_{k-1}} \\
&- \cdots - (b_{p-1} - b_p)\frac{1}{\prod_{s=0}^{p-2} x_{k-s}} - b_p \frac{1}{\prod_{s=0}^{p-1} x_{k-s}}
\end{aligned} \tag{33}$$

Therefore, it follows that for $k \geq p+1$, $x_k \leq b_1$.

Hence, $\bar{x} := \overline{\lim}_{k\to\infty} x_k \in [1, b_1]$. By (33) we have

$$\begin{aligned}
&\overline{\lim_{k\to\infty}}\, x_{k+1} \\
&\leq \overline{\lim_{k\to\infty}} \Bigg(b_1 - (b_1 - b_2)\frac{1}{x_k} - (b_2 - b_3)\frac{1}{x_k x_{k-1}} \\
&\qquad - \cdots - (b_{p-1} - b_p)\frac{1}{\prod_{s=0}^{p-2} x_{k-s}} - b_p \frac{1}{\prod_{s=0}^{p-1} x_{k-s}}\Bigg) \\
&\leq b_1 - (b_1 - b_2)\frac{1}{\overline{\lim}_{k\to\infty} x_k} - (b_2 - b_3)\frac{1}{\overline{\lim}_{k\to\infty} x_k x_{k-1}} \\
&\quad - \cdots - (b_{p-1} - b_p)\frac{1}{\overline{\lim}_{k\to\infty} \prod_{s=0}^{p-2} x_{k-s}} - b_p \frac{1}{\overline{\lim}_{k\to\infty} \prod_{s=0}^{p-1} x_{k-s}}
\end{aligned}$$

That is

$$\begin{aligned}
\bar{x} \leq\ & b_1 - (b_1 - b_2)\frac{1}{\bar{x}} - (b_2 - b_3)\frac{1}{\bar{x}^2} \\
&- \cdots - (b_{p-1} - b_p)\frac{1}{\bar{x}^{p-1}} - b_p \frac{1}{\bar{x}^p}
\end{aligned}$$

So $P(\bar{x}) \leq 0$, which contradicts to (14). Hence the sufficiency is proved.

The necessity part follows in a similar style as that of the one parameter case (Li and Xie, 2004) and is omitted here. □

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REFERENCES

Guo, L. (1997). On critical stability of discrete-time adaptive nonlinear control. *IEEE Trans. Autom. Contr.* **42**, 1488–1499.

Guo, L. and C. Wei (1996). LS-based discrete-time adaptive nonlinear control: Feasability and limitations. *Science in China* **39**(3), 255–269.

Kanellakopoulos, I. (1994). A discrete-time adaptive nonlinear system. *IEEE Trans. Autom. Contr.* **39**, 2362–2364.

Li, C. and L.-L. Xie (2004). On robust stability of discrete-time adaptive nonlinear control. *Systems & Control Letters*. Submitted. Also see http://lsc.amss.ac.cn/∼xie/.

Mitrinovic, D. S. (1970). *Analytic Inequalities.* Springer-Verlag.

Xie, L.-L. and L. Guo (1999). Fundamental limitations of discrete-time adaptive nonlinear control. *IEEE Trans. Autom. Contr.* **44**, 1777–1782.

Xie, L.-L. and L. Guo (2000*a*). Adaptive control of discrete-time nonlinear systems with structural uncertainties. In: *Lectures on Systems, Control, and Information.* AMS/IP. pp. 49–90.

Xie, L.-L. and L. Guo (2000*b*). How much uncertainty can be dealt with by feedback?. *IEEE Trans. Autom. Contr.* **45**, 2203–2217.

Zhao, J. and I. Kanellakopoulos (2002). Active identification for discrete-time nonlinear control—Part I: output-feedback systems. *IEEE Trans. Autom. Contr.* **47**, 210–224.