Additional Problems for Discussion

Example 1. Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first two successes. Find the joint mass function of X_1 and X_2.

Solution:
Since the trials are independent of each other, X_1 is independent of X_2.

$$\begin{aligned} P(X_1 = m, X_2 = n) &= P(X_1 = m)P(X_2 = n) \\ &= (1-p)^m p(1-p)^n p \\ &= p^2(1-p)^{m+n} \end{aligned}$$

Example 2. The joint probability density function of X and Y is given by

$$f(x, y) = e^{-(x+y)}, \quad 0 \le x < \infty, 0 \le y < \infty$$

Find (a) $P\{X < Y\}$ and $P\{X < a\}$.

Solution:
Compute marginal density first

$$\begin{aligned} f_X(x) &= \int_y e^{-(x+y)} dy \\ &= e^{-x} \\ f_Y(y) &= e^{-y} \end{aligned}$$

(a)

$$\begin{aligned} P(X < Y) &= \int_Y P(X < y | Y = y) f_Y(y) dy \\ &= \int_Y [\int_0^y f_X(x)dx] f_Y(y) dy \\ &= \int_Y [1 - e^{-y}] e^{-y} dy \\ &= 1/2 \end{aligned}$$

Remark: X is independent of Y in this case because $f_{X,Y}(x, y) = f_X(x) f_Y(y)$.

In general, if they are not independent, the problem should be solved as follows:

$$\begin{aligned} P(X<Y) &= \int_Y P(X<y|Y=y)f_Y(y)dy \\ &= \int_Y \int_0^y f_{X|Y}(x,y)f_Y(y)dxdy \\ &= \int_Y \int_0^y f_{X,Y}(x,y)dxdy \\ &= \int_Y \int_0^y f_X(x)f_Y(y)dxdy \qquad IF\ X\ IS\ INDEPENDENT\ OF\ Y \end{aligned}$$

(b)

$$\begin{aligned} P(X<a) &= \int_0^a e^{-x}dx \\ &= 1-e^{-a} \end{aligned}$$

Example 3. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)}, & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? What if $f(x,y)$ were given by

$$f(x,y) = \begin{cases} 2, & 0<x<y, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:
(a)

$$\begin{aligned} f(x) &= \int_y xe^{-(x+y)}dy \\ &= xe^{-x} \\ f(y) &= \int_x xe^{-(x+y)}dx \\ &= e^{-y} \\ f(x,y) &= f(x)f(y) \qquad INDEPENDENT \end{aligned}$$

(b)

$$\begin{aligned}
f(x) &= \int_0^1 2dy \\
&= 2 \\
f(y) &= \int_0^y 2dx \\
&= 2y \\
f(x,y) &\neq f(x)f(y) \qquad DEPENDENT
\end{aligned}$$

Example 4. The joint density of X and Y is

$$f(x,y) = \begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?
(b) Find the density function of X.
(c) Find $P\{X+Y<1\}$.

Solution:
(a) and (b)

$$\begin{aligned}
f(x) &= \int_0^1 (x+y)dy \\
&= x+1/2 \\
f(y) &= \int_0^1 (x+y)dx \\
&= y+1/2 \\
f(x,y) &\neq f(x)f(y) \qquad DEPENDENT
\end{aligned}$$

(c) Let $Z = X+Y$ and the distribution of Z is given in the lecture notes by using Jacobian transformation with $v = x$ and $z = x + y$.

$$\begin{aligned}
f_Z(z) &= \int_v f_{X,Y}(v, z-v)dv \\
&= \int_v z dv \\
&= \begin{cases} \int_0^z z dv, & z \le 1; \\ \int_{z-1}^1 z dv, & z > 1. \end{cases} \\
&= \begin{cases} z^2, & z \le 1; \\ 2z - z^2, & z > 1. \end{cases} \\
P(Z < 1) &= \int_0^1 z^2 dz \\
&= 1/3
\end{aligned}$$

Example 5. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Also compute $P\{X_1 < X_2\}$.

Solution:
(a)From assignment 4 question 8, we have

$$\begin{aligned}
p_Z(z) &= \int_v p_{X_1}(v) p_{X_2}(\frac{v}{z}) |\frac{v}{z^2}| dv \\
&= \int_0^\infty \lambda_1 e^{-\lambda_1 v} \lambda_2 e^{-\lambda_2 \frac{v}{z}} |\frac{v}{z^2}| dv \\
&= \frac{\lambda_1 \lambda_2}{z^2} \int_0^\infty v e^{-v(\lambda_1 + \frac{\lambda_2}{z})} dv \\
&= \frac{\lambda_1 \lambda_2}{z^2} \left[-\frac{1}{\lambda_1 + \lambda_2/z} v e^{-v(\lambda_1 + \frac{\lambda_2}{z})} |_0^\infty - \int_0^\infty -\frac{1}{\lambda_1 + \lambda_2/z} e^{-v(\lambda_1 + \frac{\lambda_2}{z})} dv \right] \\
&= \frac{\lambda_1 \lambda_2}{z^2} \frac{1}{\lambda_1 + \lambda_2/z} \int_0^\infty e^{-v(\lambda_1 + \frac{\lambda_2}{z})} dv \\
&= \frac{\lambda_1 \lambda_2}{z^2} (\frac{1}{\lambda_1 + \lambda_2/z})^2
\end{aligned}$$

(b)

$$\begin{aligned}
P(X_1 < X_2) &= \int_0^\infty P(X_1 < x_2 | X_2 = x_2) f(x_2) dx_2 \\
&= \int_0^\infty [\int_0^{x_2} f(x_1) dx_1] f(x_2) dx_2 \\
&= \int_0^\infty (1 - e^{-\lambda_1 x_2}) \lambda_2 e^{-\lambda_2 x_2} dx_2 \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} - \int_0^\infty \lambda_2 e^{(\lambda_1 + \lambda_2) x_2} dx_2 \\
&= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}$$

Example 6. Show that the jointly continuous (discrete) random variables $X_1, \dots, X_n$ are independent if and only if their probability density (mass) function $f(x_1, \dots, x_n)$ can be written as

$$f(x_1, \dots, x_n) = \prod_{i=1}^n g_i(x_i)$$

for nonnegative functions $g_i(x), i = 1, \dots, n$.

Solution:
First, we prove if $X_1, \dots, X_n$ are independent,

$$f(x_1, \dots, x_n) = \prod_{i=1}^n g_i(x_i).$$

Since $X_1, \dots, X_n$ are independent, we have

$$f(x_1, \dots, x_n) = \prod_{i=1}^n f_{X_i}(x_i)$$

Let $g_i(x_i) = f_{X_i}(x_i)$, we prove the claim of the problem.

Next, we prove if

$$f(x_1, \dots, x_n) = \prod_{i=1}^n g_i(x_i).$$

then $X_1, \dots, X_n$ are independent.

Let

$$C_i = \int_{-\infty}^\infty g_i(x) dx, \ i = 1, \dots, n$$

Since the n-fold integral of the joint density function is equal to 1, we obtain that

$$\prod_{i=1}^n \int_{-\infty}^\infty g_i(x) dx = \prod_{i=1}^n C_i = 1$$

Integrating the joint density over all x_i except x_j gives that

$$f_{X_i}(x_i) = g_j(x_j) \prod_{i \neq j} C_i = \frac{g_j(x_j)}{C_j}$$

Therefore,

$$f(x_1, \ldots, x_n) = \frac{\prod_{i=1}^n g_i(x_i)}{\prod_{i=1}^n C_i = 1} = \prod_{i=1}^n f_{X_i}(x_i)$$

which shows that the random variables are independent.
Example 7. Let X_1, X_2, X_3 be independent and identically distributed continuous random variables. Compute

$$\begin{aligned}
&(a)\ P\{X_1 > X_2 | X_1 > X_3\};\\
&(b)\ P\{X_1 > X_2 | X_1 < X_3\};\\
&(c)\ P\{X_1 > X_2 | X_2 > X_3\};\\
&(d)\ P\{X_1 > X_2 | X_2 < X_3\}.
\end{aligned}$$

Solution:

$$\begin{aligned}
&(a)\ P\{X_1 > X_2 | X_1 > X_3\} = \frac{P\{X_1 = \max(X_1, X_2, X_3)\}}{P\{X_1 > X_3\}} = \frac{1/3}{1/2} = \frac{2}{3}\\
&(b)\ P\{X_1 > X_2 | X_1 < X_3\} = \frac{P\{X_3 > X_1 > X_2\}}{P\{X_1 < X_3\}} = \frac{1/3!}{1/2} = \frac{1}{3}\\
&(c)\ P\{X_1 > X_2 | X_2 > X_3\} = \frac{P\{X_1 > X_2 > X_3\}}{P\{X_2 > X_3\}} = \frac{1/3!}{1/2} = \frac{1}{3}\\
&(d)\ P\{X_1 > X_2 | X_2 < X_3\} = \frac{P\{X_2 = \min(X_1, X_2, X_3)\}}{P\{X_2 < X_3\}} = \frac{1/3}{1/2} = \frac{2}{3}
\end{aligned}$$

We calculate $P\{X_1 > X_3\}$, $P\{X_1 = \max(X_1, X_2, X_3)\}$, and $P\{X_1 > X_2 > X_3\}$ here, others can be obtained in a similar way. Since X_1, X_2, X_3 be independent and identically distributed, assume probability density function and probability distribution function of $X_i, i = 1, 2, 3$ to be $f(x)$ and $F(x)$ respectively, then we obtain that

$$\begin{aligned}
P\{X_1 > X_3\} &= \int_{-\infty}^{\infty} P\{X_1 > t | X_3 = t\} f(t) dt = \int_{-\infty}^{\infty} [1 - F(t)] dF(t)\\
&= 1 - \frac{1}{2} F^2(t) \Big|_{-\infty}^{\infty} = \frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
P\{X_1 = \max(X_1, X_2, X_3)\} &= P\{X_1 \geq X_2, X_1 \geq X_3)\} \\
&= \int_{-\infty}^{\infty} P\{X_2 \leq t, X_3 \leq t | X_1 = t\} f(t) dt \\
&= \int_{-\infty}^{\infty} F^2(t) dF(t) = \frac{1}{3} F^3(t) \Big|_{-\infty}^{\infty} = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
P\{X_1 > X_2 > X_3\} &= \int_{-\infty}^{\infty} P\{t > X_2 > X_3 | X_1 = t\} f(t) dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{t} P\{t > X_2 > s | X_1 = t, X_3 = s\} f(t) f(s) dt ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{t} [F(t) - F(s)] dF(s) dF(t) \\
&= \int_{-\infty}^{\infty} \frac{1}{2} F^2(t) dF(t) = \frac{1}{2} \cdot \frac{1}{3}
\end{aligned}
$$