ECE 316-Solutions of Problem Set 5

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Solution 1

Our estimate of X is $\widehat{X} = aY + b$ where a and b should be chosen to minimize the mean squared error which is $J = E(X - \widehat{X})^2$. To minimize J, we set $\frac{\partial J}{\partial a} = 0$ and $\frac{\partial J}{\partial b} = 0$

Now:

$$J = E(X - aY - b)^2 = E[X^2] - 2aE[XY] - 2bE[X] - +2abE[Y] + a^2E[Y^2] + b^2$$

Therefore,

$$\begin{aligned} \frac{\partial J}{\partial a} &= 0 \\ \Rightarrow E(2(X - (aY + b))(-Y)) &= 0 \\ \Rightarrow E(XY) &= aE(Y^2) + bE(Y) \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{\partial J}{\partial b} &= 0 \\ \Rightarrow E((2(X - (aY + b))(-1)) &= 0 \\ \Rightarrow E(X) = aE(Y) + b \end{aligned} \tag{2}$$

Solving 1 and 2 for a and b

$$\begin{aligned} a &= \frac{E(XY) - E(X)E(Y)}{E(Y^2) - E(Y)^2} \\ &= \frac{cov(X,Y)}{Var(Y)} \end{aligned}$$

and

$$b = E(X) - aE(Y)$$

Problem 2

$$\begin{aligned} f_X(z) &= \frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2}) \qquad -\infty < z < \infty \\ E(Z) &= E[X\mathbf{1}_{[X>x]}] = \int_x^\infty z\frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})dz \\ &= -\frac{1}{\sqrt{2\pi}}\exp(-\frac{z^2}{2})|_x^\infty \\ &= \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2}) \end{aligned}$$

Problem 3

This is an important result.

First note by definition:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = h(x)\int_{-\infty}^{\infty} g(y)dy = C_1 h(x)$$

where $C_1 = \int_{-\infty}^{\infty} g(y)dy$

Similarly, we obtain

$$f_Y(y) = C_2 g(y)$$

where $C_2 = \int_{-\infty}^{\infty} h(x)dx$.

On the other hand we know

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = \int_{-\infty}^{\infty} h(x)dx\int_{-\infty}^{\infty} g(y)dy = C_2C_1$$

Therefore $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ or the random variables are independent.

Now consider the joint density given in the example, then it can be written as:

$$f_{X,Y}(x,y) = Cxy\mathbf{1}_{[0<x<1]}\mathbf{1}_{[0<y<1]}\mathbf{1}_{[0<x+y<1]}, \quad -\infty < x < \infty;\ -\infty < y < \infty$$

which clearly cannot be written as $h(x)g(y)$ for some functions $h(.)$ and $g(.)$ because the last indicator function introduces dependency because it involves both x and y.

Therefore $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ and so the random variables cannot be independent.

Problem 4

Instead of solving this problem for three random variables we can solve it for the case of n independent random variables.

$X_1, \cdots X_n$ are i.i.d r.v having uniform distribution over $(0,1)$ Let $Y = \max(X_1, \cdots X_n)$, $Z = \min(X_1, \cdots X_n)$

$$
\begin{aligned}
P(Y \leq y) &= P(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y) = P(X_1 \leq y)P(X_2 \leq y) \cdots P(X_n \leq y) \quad \textit{by independence} \\
&= \begin{cases} 0, & y < 0 \\ y^n, & 0 < y < 1 \\ 1, & y > 1 \end{cases}
\end{aligned}
$$

$$f_Y(y) = ny^{n-1} 0 < y < 1$$

$$
\begin{aligned}
P(Z \leq z) &= 1 - P(Z > z) \\
&= 1 - P(X_1 > z)P(X_2 > z) \cdots P(X_n > z) \\
&= \begin{cases} 1, & y < 0 \\ 1 - (1-z)^n, & 0 < z < 1 \\ 0, & z > 1 \end{cases}
\end{aligned}
$$

$$f_Z(z) = n(1-z)^{n-1} \quad 0 < z < 1$$

$$E(Y) = \frac{n}{n+1}$$

$$E(Z) = \frac{1}{n+1}$$

So we see that as $n \to \infty$ $E[Y] \to 1$ and $E[Z] \to 0$ i.e. as the number of random variables increases the largest value we obtain is closer and closer to 1 the maximum a given r.v can be and the minimum goes towards 0 which is the minimum value a given r.v. can have.

Problem 5

$$
\begin{aligned}
E(X) &= \int_{y=0}^{\infty}\int_{x=0}^{\infty} x\frac{1}{y}\exp(-(y+x/y))dxdy \\
&= \int_{y=0}^{\infty}\frac{1}{y}\exp(-y)\int_{x=0}^{\infty} x\exp(-x/y)dxdy \\
&= \int_{y=0}^{\infty}\frac{1}{y}\exp(-y)y^2dy \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E(Y) &= \int_{y=0}^{\infty}\int_{x=0}^{\infty} y\frac{1}{y}\exp(-(y+x/y))dxdy \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E(XY) &= \int_{y=0}^{\infty}\int_{x=0}^{\infty} xy\frac{1}{y}\exp(-(y+x/y))dxdy \\
&= \int_{y=0}^{\infty} y\frac{1}{y}\exp(-y)\int_{x=0}^{\infty} x\exp(-x/y)dxdy \\
&= \int_{y=0}^{\infty}\exp(-y)y^2dy \\
&= 2 \\
Cov(X,Y) &= E(XY)-E(X)E(Y) = 1
\end{aligned}
$$

Problem 6

X_1, X_2, X_3, X_4 are pairwise uncorrelated random variables each having mean 0 and variance 1.

$$
\begin{aligned}
cov(X_1+X_2, X_2+X_3) &= cov(X_1,X_2)+cov(X_1,X_3)+cov(X_2,X_2)+cov(X_2,X_3) \\
&= 0+0+1+0 = 1 \\
var(X_1+X_2) &= var(X_1)+var(X_2)+2cov(X_1,X_2) \\
&= 1+1+0 = 2 \\
var(X_2+X_3) &= var(X_2)+var(X_3)+2cov(X_2,X_3) \\
&= 1+1+0 = 2 \\
\rho(X_1+X_2, X_2+X_3) &= \frac{cov(X_1+X_2, X_2+X_3)}{\sqrt{var(X_1+X_2)var(X_2+X_3)}} \\
&= 0.5 \\
cov(X_1+X_2, X_3+X_4) &= cov(X_1,X_3)+cov(X_1,X_4)+cov(X_2,X_3)+cov(X_2,X_4) \\
&= 0 \\
\rho(X_1+X_2, X_3+X_4) &= 0
\end{aligned}
$$

Problem 7

$$
\begin{aligned}
f(x,y) &= \frac{e^{-x/y}e^{-y}}{y} \qquad 0 < x < \infty, 0 < y < \infty \\
f_Y(y) &= \int_{x=0}^{\infty} \frac{e^{-x/y}e^{-y}}{y} dx \\
&= e^{-y} \quad y > 0 \\
f_{X|Y}(x|y) &= \frac{f(x,y)}{f(y)} = \frac{e^{-x/y}}{y} \\
E(X^2|Y=y) &= \int_{x=0}^{\infty} x^2 \frac{e^{-x/y}}{y} dx \\
&= 2y^2
\end{aligned}
$$

Problem 8

This is the same argument as in Problem 1 with the linear term in Y missing.

Let $J = E((X-a)^2)$. To minimize J, we should have $\frac{dJ}{da} = 0$. Therefore

$$
\begin{aligned}
\frac{dJ}{da} &= 0 \\
\Rightarrow 2E(X-a) &= 0 \\
\Rightarrow a &= E(X)
\end{aligned}
$$

The second part needs some proof.

First note that $|X-a| = (X-a)\mathbf{1}_{[X>a]} - (X-a)\mathbf{1}_{[X\leq a]}$.

Therefore:

$$E|X-a| = \int_a^{\infty} (x-a) f_X(x) dx - \int_{-\infty}^{a} (x-a) f_X(x) dx$$

Differentiating (using Liebniz's rule) w.r.t a and setting the derivative to 0 gives:

$$-\int_a^{\infty} f_X(x) dx + \int_{-\infty}^{a} f_X(x) dx = 0$$

Noting that $\int_{-\infty}^{a} f_X(x)dx = F_X(a)$ and $\int_a^{\infty} f_X(x)dx = 1 - F_X(a)$ we obtain:

$$F_X(a) = 1 - F_X(a)$$

or $F_X(a) = 0.5$ so a corresponds to the median.

Problem 9

Without loss of generality assume $E[X] = E[Y] = 0$ then

$$Cov(X+Y, X-Y) = E[(X-Y)(X+Y)] = E[X^2] - E[Y^2] = (Var(X) - Var(Y) = 0$$

since X and Y are identically distributed.

Problem 10

Without loss of generality assume $E[X] = E[Y] = 0$.

Assume $E[Y^2] > 0$ and let $a = \frac{E[XY]}{E[Y^2]}$.

Then

$$\begin{aligned} 0 &\leq E[Y^2]E[(X - aY)^2] \\ &= E[Y^2](E[X^2] + a^2E[Y^2] - 2aE[XY] \\ &= E[Y^2]E[X^2] - (E[XY])^2 \end{aligned}$$

from which the result follows.

Problem 11

Once again without loss of generality assume that $E[X] = E[Y] = 0$ (convince yourselves that if $E[X] = m_X$ and $E[Y] = m_Y$ the answer still holds. Now $Y = a + bX$,

Hence $cov(XY) = E(XY) = aE[X] + bE[X^2] = bvar(X)$

Now $var(Y) = b^2Var(X)$

Therefore

$$\rho(X, Y) = \frac{bvar(X)}{|b|var(X)} = sign(b)$$

from which the answer follows.