ECE 316-Solutions of Problem Set 3

Solution 1

$$
\begin{aligned}
P(A_1A_2\cdots A_n) &= P(A_1A_2\cdots A_{n-1})P(A_n|A_1A_2\cdots A_{n-1}) \\
&= P(A_1A_2\cdots A_{n-2})P(A_{n-1}|A_1A_2\cdots A_{n-2})P(A_n|A_1A_2\cdots A_{n-1}) \\
&\vdots \\
&= P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\ldots P(A_n|A_1A_2\cdots A_{n-1}) \\
&= P(A_1)P(A_2|A_1)P(A_3|A_2)\cdots P(A_{i+1}|A_i)\cdots P(A_n|A_{n-1})
\end{aligned}
$$

using the given property

Second Part

For the three events A_k, A_j and A_l with $k < j < l$, we can write

$$
\begin{aligned}
& P(A_l|A_jA_k) = P(A_l|A_j) \\
\implies & \frac{P(A_lA_jA_k)}{P(A_jA_k)} = \frac{P(A_lA_j)}{P(A_j)} \\
\implies & \frac{P(A_lA_jA_k)}{P(A_j)} = \frac{P(A_jA_k)}{P(A_j)}\frac{P(A_lA_j)}{P(A_j)} \\
\implies & P(A_kA_l|A_j) = P(A_k|A_j)P(A_l|A_j)
\end{aligned}
$$

Solution 2

Let A and B are independent events. So we have

$$P(AB) = P(A)P(B).$$

We need to show that A and B^c are independent, i.e. $P(A)P(B^c) = P(A)P(B^c)$
Since $A = AB \cup AB^c$, and AB and AB^c are mutually exclusive, we can write

$$
\begin{aligned}
& P(A) = P(AB) + P(AB^c) \\
\implies & P(A) = P(A)P(B) + P(AB^c) \\
\implies & P(A)(1 - P(B)) = P(AB^c) \\
\implies & P(A)P(B^c) = P(AB^c) \text{ Hence } A \text{ and } B^c \text{ are independent}
\end{aligned}
$$

Second Part

$$\begin{aligned} P(A|B) &= p(A|B^c) \\ \implies \frac{P(AB)}{P(B)} &= \frac{P(AB^c)}{P(B^c)} \\ \implies P(AB)(1-P(B)) &= P(AB^c)P(B) \\ \implies P(AB) &= (P(AB)+P(AB^c))P(B) \\ \implies P(AB) &= P(A)P(B) \end{aligned}$$

Hence A and B are independent

Solution 3

$$\begin{aligned} F(x) = \mathbf{Pr}\{X(\omega) \leq x\} &= 1 - e^{-2x} \quad ; x \geq 0 \\ &= 0 \quad otherwise \end{aligned}$$

a)

$$\begin{aligned} &\mathbf{Pr}.\{X(\omega) \leq 1\} = F(1) = 1 - e^{-2} \\ &\mathbf{Pr}.\{X(\omega) > 2\} = 1 - \mathbf{Pr}.\{X(\omega) \leq 2\} \\ &\qquad\qquad\qquad\quad = 1 - F(2) = 1 - (1 - e^{-4}) = e^{-4} \\ &\mathbf{Pr}.\{X(\omega) = 3\} = 0 \quad \text{because } X(w) \text{ is a continuous random variable} \end{aligned}$$

b)

$$\begin{aligned} p_X(x) = \frac{d}{dx}\left(F_X(x)\right) &= 2e^{-2x}, \quad x \geq 0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$

c)

$$P_Y(Y(\omega) = 0) = P(X(\omega) \leq 2) = 1 - e^{-4}$$

Solution 4

Let $T(\omega)$ be a geometrically distributed random variable, i.e.,

$$Pr(T = k) = pq^{k-1}, \quad p = 1 - q$$

Left Hand Side

$$\begin{aligned}
Pr(T > n_0 + k | T > n_0) &= \frac{Pr(T > n_0 + k \cap T > n_0)}{Pr(T > n_0)} \\
&= \frac{Pr(T > n_0 + k)}{Pr(T > n_0)} \\
&= \frac{\sum\limits_{j=n_0+k+1}^{\infty} pq^{j-1}}{\sum\limits_{j=n_0+1}^{\infty} pq^{j-1}} \\
&= \frac{q^{n_0+k}}{q^{n_0}} \\
&= q^k
\end{aligned}$$

Right Hand Side

$$\begin{aligned}
Pr(T > k) &= \sum_{j=k+1}^{\infty} pq^{j-1} \\
&= p\frac{q^k}{1-q} \\
&= p\frac{q^k}{p} \\
&= q^k
\end{aligned}$$

Hence Left Hand Side=Right Hand Side

Solution 5

Let $X(\omega)$ is a Poisson random variable with parameter λ.
Therefore

$$P(X(\omega) = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

$$
\begin{aligned}
E(X^2) &= \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} k \frac{e^{-\lambda}\lambda^k}{(k-1)!} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{k!} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{d}{d\lambda} \left(\frac{\lambda^k}{(k-1)!} \right) \\
&= \lambda e^{-\lambda} \frac{d}{d\lambda} \sum_{k=1}^{\infty} \left(\frac{\lambda^k}{(k-1)!} \right) \\
&= \lambda e^{-\lambda} \frac{d}{d\lambda} (\lambda e^{\lambda}) \\
&= \lambda e^{-\lambda} (\lambda e^{\lambda} + e^{\lambda}) \\
&= \lambda^2 + \lambda
\end{aligned}
$$

$$
Var(X) = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda
$$

Solution 6

$$
\begin{aligned}
E(X^2) + E(X) &= \sum_{k=0}^{\infty} k^2 P(X = k) + \sum_{k=0}^{\infty} k P(X = k) \\
&= \sum_{k=1}^{\infty} k^2 P(X = k) + \sum_{k=1}^{\infty} k P(X = k) \\
&= \sum_{k=1}^{\infty} k^2 (P(X \geq k) - P(X > k)) + \sum_{k=1}^{\infty} k (P(X \geq k) - P(X > k)) \\
&= \sum_{k=1}^{\infty} k^2 P(X \geq k) - \sum_{k=1}^{\infty} k^2 P(X > k) + \sum_{k=1}^{\infty} k P(X \geq k) - \sum_{k=1}^{\infty} k P(X > k) \\
&= \sum_{k=1}^{\infty} ((k-1)^2 + 2k - 1) P(X \geq k) - \sum_{k=1}^{\infty} k^2 P(X > k) + \sum_{k=1}^{\infty} (k - 1 + 1) P(X \geq k) \\
&\quad - \sum_{k=1}^{\infty} k P(X > k) \\
&= \sum_{k=1}^{\infty} (k-1)^2 P(X \geq k) + \sum_{k=1}^{\infty} (2k-1) P(X \geq k) - \sum_{k=1}^{\infty} k^2 P(X > k) + \sum_{k=1}^{\infty} (k-1) P(X \geq k) \\
&\quad + \sum_{k=1}^{\infty} P(X \geq k) - \sum_{k=1}^{\infty} k P(X > k) \\
&= \sum_{k=1}^{\infty} 2k P(X \geq k) \\
&\text{since} \quad \sum_{k=1}^{\infty} (k-1)^2 P(X \geq k) = \sum_{k=1}^{\infty} k^2 P(X > k) \\
&\text{and} \quad \sum_{k=1}^{\infty} (k-1) P(X \geq k) = \sum_{k=1}^{\infty} k P(X > k)
\end{aligned}
$$

Therefore

$$
E(X^2) = \sum_{k=1}^{\infty} 2k P(X \geq k) - E(X)
$$

Solution 7

$$
\begin{aligned}
E(X) &= \int_0^\infty x d(F(x)) \\
&= -\int_0^\infty x d(1-F(x)) \\
&= -x(1-F(x))|_0^\infty + \int_0^\infty (1-F(x))dx \quad \text{integrating by parts} \\
&= \int_0^\infty (1-F(x))dx
\end{aligned}
$$

If $\quad -\infty < X < \infty, \quad$ Then

$$
\begin{aligned}
E(X) &= \int_{-\infty}^\infty x d(F(x)) \\
&= \int_{-\infty}^0 x d(F(x)) + \int_0^\infty x d(F(x)) \\
&= \int_{-\infty}^0 x d(F(x)) + \int_0^\infty (1-F(x))dx \\
&= xF(x)|_{-\infty}^0 - \int_{-\infty}^0 F(x)dx + \int_0^\infty (1-F(x))dx \\
&= -\int_{-\infty}^0 F(x)dx + \int_0^\infty (1-F(x))dx
\end{aligned}
$$

Solution 8

The properties of pdf are
1. non-negative
2. $\int p(x)dx = 1$

a) If $c > 0$, then $p(x) = c(x - x^2)$ is non-negative when $0 \leq x \leq 1$. Therefore

$$0 \leq a < b \leq 1$$

If $c < 0$, then $p(x) = c(x - x^2)$ is non-negative when $x \leq 0$ or $x \geq 1$. Therefore $a < b \leq 0$ or $1 \leq a < b$

b)

$$\int p(x) = 1$$

$$\implies \int_a^b c(x - x^2)dx = 1$$

$$\implies c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_a^b = c\left[\frac{b^2}{2} - \frac{b^3}{3} - \frac{a^2}{2} + \frac{a^3}{3}\right] = 1$$

$$\implies c = \frac{1}{\frac{1}{2}(b^2 - a^2) + \frac{1}{3}(a^3 - b^3)}$$

Solution 9

X is a non-negative random variable

$$E(X) = \int_0^\infty (1 - F(x))dx = \int_0^\infty P(X \geq x)dx = \sum_{k=1}^{\infty} \int_{k-1}^{k} P(X \geq x)dx$$

Since $P(X \geq x)$ is a non-increasing function,

$$\int_{k-1}^{k} P(X \geq k)dx \leq \int_{k-1}^{k} P(X \geq x)dx \leq \int_{k-1}^{k} P(X \geq k-1)dx$$

$$\implies P(X \geq k) \times 1 \leq \int_{k-1}^{k} P(X \geq x)dx \leq P(X \geq k-1) \times 1$$

$$\implies \sum_{k=1}^{\infty} P(X \geq k) \leq E(x) \leq \sum_{k=1}^{\infty} P(X \geq k-1)$$

$$\implies \sum_{k=1}^{\infty} P(X \geq k) \leq E(x) \leq P(X \geq 0) + \sum_{k=1}^{\infty} P(X \geq k)$$

Since $P(X >= 0) = 1$

$$\implies \sum_{k=1}^{\infty} P(X \geq k) \leq E(x) \leq 1 + \sum_{k=1}^{\infty} P(X \geq k)$$

Solution 10

$$
\begin{aligned}
E\left[\frac{1}{X}\right] &= \sum_{k=1}^{\infty}\frac{1}{k}p(1-p)^{k-1} \\
&= \frac{p}{1-p}\sum_{k=1}^{\infty}\frac{(1-p)^k}{k} \\
&= \frac{p}{1-p}\sum_{k=1}^{\infty}\int_0^{1-p} x^{k-1}dx \\
&= \frac{p}{1-p}\int_0^{1-p}\sum_{k=1}^{\infty} x^{k-1}dx \\
&= \frac{p}{1-p}\int_0^{1-p}\frac{1}{1-x}dx \\
&= \frac{p}{1-p}[-log(1-x)]_0^{1-p} \\
&= -\frac{p\ logp}{1-p}
\end{aligned}
$$

Solution 11

a) $E(X^2) = p$ and $E(X) = p$. Hence

$$Var(X) = E(X^2) - E(X)^2 = p - p^2$$

b) Since X can take values 0 and 1, Y can take values a and b

$$P(Y = b) = P(X = 0) = 1 - p$$

$$P(Y = a) = P(X = 1) = p$$

$$E(Y) = ap + b(1-p)$$

$$E(Y^2) = a^2p + b^2(1-p)$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = (a-b)^2p(1-p)$$