ECE 316-Solutions of Problem Set 4

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Solution 1

a) Let us denote by $E_i,\ i = 1, 2, ..., n$ the event that the i^{th} name matches with the corresponding address. We need to find out $P\left(\bigcup_{i=1}^{n} E_i\right)$ and by the inclusion-exclusion principle, it can be written as

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1<i_2<\cdots<i_n} P(E_{i_1}E_{i_2}\cdots E_{i_r})$$
$$+ \cdots + (-1)^{n+1} P(E_1E_2\cdots E_n)$$

Now there are $n!$ possible ways of matching the names with the address. Furthermore, $E_{i_1}Ei_2...Ei_r$, the event that each of the r names $i_1, i_2, ...i_r$ matches with the address, can occur in any of $(n-r)(n-r-1)\cdots 3.2.1 = (n-r)!$ possible ways; for of the remaining $n-r$ names, the first name can match with any of the $n-r$ addresses, the second can match with any of the remaining $n-r-1$ addresses and so on. Hence,

$$P(E_{i_1}E_{i_2}\cdots E_{i_r}) = \frac{(n-r)!}{n!}$$

Also there are $\binom{n}{r}$ terms in $\sum\limits_{i_1<i_2\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r})$, we see that

$$\sum_{i_1<i_2\cdots<i_r} P(E_{i_1}E_{i_2}\cdots E_{i_r}) = \frac{n!(n-r)!}{(n-r)!r!n!} = \frac{1}{r!}$$

Therefore

$$P\left(\bigcup_{i=1}^{n}\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}$$

which for $n- > \infty$ converges to $1 - e^{-1}$ since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

b) Let N denote the number of bits transmitted. Given that the number of bits transmitted within a given time interval is Poisson with parameter λ. Therefore, within that time interval,

$$P(N = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

Let N_1 and N_0 denote the number of 1's and 0's transmitted.

$$\begin{aligned}
P(N_1 = m, N_0 = n) &= \sum_{k=0}^{\infty} P(N_1 = m, N_0 = n | N = k) P(N = k) \\
&= P(N_1 = m, N_0 = n | N = m+n) P(N = m+n) \\
&= \frac{(m+n)!}{m!n!} p^m (1-p)^n e^{-\lambda} \frac{\lambda^{m+n}}{(m+n)!} \\
&= e^{-\lambda p} \frac{(\lambda p)^m}{m!} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^n}{n!}
\end{aligned}$$

Therefore

$$\begin{aligned}
P(N_1 = m) &= \sum_{n=0}^{\infty} e^{-\lambda p} \frac{(\lambda p)^m}{m!} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^n}{n!} \\
&= e^{-\lambda p} \frac{(\lambda p)^m}{m!}
\end{aligned}$$

and

$$\begin{aligned}
P(N_0 = n) &= \sum_{m=0}^{\infty} e^{-\lambda p} \frac{(\lambda p)^m}{m!} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^n}{n!} \\
&= e^{-\lambda(1-p)} \frac{(\lambda(1-p))^n}{n!}
\end{aligned}$$

Therefore, the number of 1's transmitted is Poisson with parameter $p\lambda$ and the number of 0's transmitted is Poisson with parameter $(1-p)\lambda$

c) The solution to this problem just follows from the definition of conditional probabilities.

Indeed

$$\begin{aligned}
p_{X,Y,Z}(x,y,z) &= p_{Z|X,Y}(z|x,y) p_{X,Y}(x,y) \\
&= p_{Z|X,Y}(z|x,y) p_{Y|X}(y|x) P_X(x) \quad (\textit{by substituting for } p_{X,Y}(x,y))
\end{aligned}$$

Solution 2

X is uniformly distributed on [c,d]. Therefore

$$\begin{aligned}
p_X(x) &= \frac{1}{d-c}, \quad x \in [c,d] \\
&= 0 \qquad \text{otherwise}
\end{aligned}$$

a)

$$Y = aX + b$$

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(aX + b \le y) \\
&= P(X \le \frac{y-b}{a})
\end{aligned}$$

Therefore

$$F_Y(y) = \begin{cases} 0 & y < ac+b \\ \int_c^{\frac{y-b}{a}} \frac{1}{d-c} dx = \frac{1}{d-c}\left(\frac{y-b}{a} - c\right) & ac+b \le y \le ad+b \\ 1 & y > ad+b \end{cases}$$

$$f_Y(y) = \begin{cases} 0, & y < ac+b \\ \frac{1}{a(d-c)}, & ac+b \le y \le ad+b \\ 0, & y > ad+b \end{cases}$$

For the other three parts, assume $c, d > 0$

b)

$$Y = \frac{1}{X}$$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\frac{1}{X} \le y) \\ &= P(X \ge \frac{1}{y}) \end{aligned}$$

Therefore

$$F_Y(y) = \begin{cases} 1, & y > 1/c \\ \int_{1/y}^{d} \frac{1}{d-c} dx = \frac{d-1/y}{d-c}, & 1/d \le y \le 1/c \\ 0, & y < 1/d \end{cases}$$

So

$$f_Y(y) = \begin{cases} 0, & y > 1/c \\ \frac{1}{y^2(d-c)} & 1/d \le y \le 1/c \\ 0, & y < 1/d \end{cases}$$

c)

$$Y = X^2$$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X^2 \le y) \\ &= P(-\sqrt{y} \le X \le \sqrt{y}) \end{aligned}$$

Therefore

$$F_Y(y) = \begin{cases} 0, & y < c^2 \\ \int_c^{\sqrt{y}} \frac{1}{d-c} dx = \frac{\sqrt{y}-c}{d-c}, & c^2 \le y \le d^2 \\ 1, & y > d^2 \end{cases}$$

$$f_Y(y) = \begin{cases} 0 & y < c^2 \\ \frac{1}{2\sqrt{y}(d-c)}, & c^2 \leq y \leq d^2 \\ 0 & y > d^2 \end{cases}$$

d)

$$Y = \sqrt{X}$$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\sqrt{X} \leq y) \\ &= P(X \leq y^2) \end{aligned}$$

Therefore

$$F_Y(y) = \begin{cases} 0, & y < \sqrt{c} \\ \int_c^{y^2} \frac{1}{d-c} dx = \frac{y^2-c}{d-c}, & \sqrt{c} \leq y \leq \sqrt{d} \\ 1, & y > \sqrt{d} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 0, & y < \sqrt{c} \\ \frac{2y}{d-c}, & \sqrt{c} \leq y \leq \sqrt{d} \\ 0, & y > \sqrt{d} \end{cases}$$

Solution 3

$X \sim N(m, \sigma^2)$, $Y = e^X$.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(e^X \leq y) \\ &= P(X \leq \ln(y)) \\ &= F_X(\ln(y)) \end{aligned}$$

Therefore

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy}(F_X(\ln(y))) \\ &= f_X(\ln(y)) \frac{1}{y} \\ &= \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(y) - m)^2}{2\sigma^2}\right) \end{aligned}$$

Second Procedure

Using the approach $Y = g(X)$)

Let X be a continuous random variable having probability density function f_X. Suppose that $g(x)$ is a monotone (increasing or decreasing), differentiable (and thus continuous) function of x. Then the random variable Y defined by $Y = g(X)$ has a probability density function given by

$$f_Y(y) = \begin{cases} f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right|, & \text{if } y = g(x) \text{ for some } x \\ 0, & \text{if } y \neq g(x) \text{ for all } x \end{cases}$$

where $g^{-1}(y)$ is defined to equal that value of x such that $g(x) = y$
Now, $g(x) = e^x$ is a monotone increasing function and $g^{-1}(y) = \ln(y)$. Therefore

$$\begin{aligned} f_Y(y) &= f_X(\ln(y)) \left| \frac{d}{dy} \ln(y) \right| \\ &= \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(y) - m)^2}{2\sigma^2} \right) \end{aligned}$$

Solution 4

$$\begin{aligned} P(X + Y > z) &= P(X + Y > z | X > z) P(X > z) + P(X + Y > z | X \leq z) P(X \leq z) \\ &= 1.P(X > z) + P((X + Y > z) \cap (X \leq z)) \\ &= P(X > z) + P(X + Y > z \geq X) \end{aligned}$$

Second Part

$$\begin{aligned} E(X) &= \int_0^\infty x d(F(x)) \\ &= -\int_0^\infty x d(1 - F(x)) \\ &= -x(1 - F(x))|_0^\infty + \int_0^\infty (1 - F(x)) dx \quad \text{integrating by parts} \\ &= \int_0^\infty (1 - F(x)) dx \\ &= \int_0^\infty P(X > x) dx \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\infty P(X + Y > z \geq X) dz &= \int_0^\infty (P(X + Y > z) - P(X > z)) dz \\ &= E(X + Y) - E(X) \\ &= E(Y) \end{aligned}$$

Solution 5

First note that:

$$X^2 \leq CX$$

since $X \in [0, C]$.

Now define $a = \frac{E[X]}{C}$ and noting that $var(X) = E[X^2] - (E[X])^2$ we obtain:

$$var(X) \leq C^2 a(1-a)$$

and the r.h.s is maximized when $a = \frac{1}{2}$ to give the result.

Solution 6

Since $R = \frac{X}{Y}$ is a mapping from $\mathbb{R}^2 \to \mathbb{R}$, we need to introduce another one dimensional mapping to ensure that the Jacobian matrix is a square matrix. Otherwise Determinant J will not exist. Define mapping $g : (x, y) \to (v, r)$ with $v = x$ and $r = \frac{x}{y}$. If we solve x and y in terms of v and r, we obtain $x = v$ and $y = \frac{v}{r}$. Thus $g^{-1}(v, r) = (v, \frac{v}{r})$, and the Jacobian matrix is given by

$$J(x, y) = \begin{bmatrix} 1 & 0 \\ \frac{1}{y} & \frac{-x}{y^2} \end{bmatrix}$$

and $|det\ J| = \left|\frac{x}{y^2}\right| = \left|\frac{r^2}{v}\right|$. The density of R is given by

$$\begin{aligned} f_R(r) &= \int_v f_{X,Y}(f^{-1}(v, r)) \left|\frac{v}{r^2}\right| dv \\ &= \int_v f_{X,Y}(v, \frac{v}{r}) \left|\frac{v}{r^2}\right| dv \end{aligned}$$

If X and Y are independent, then we have

$$f_R(r) = \int_v f_X(v) f_Y(\frac{v}{r}) \left|\frac{v}{r^2}\right| dv$$

Solution 7

First we obtain the conditional density of X given $Y = y$ as:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x, y)}{f_Y(y)/y} \\ &= \frac{e^{-\frac{x}{y}} e^{-y}}{e^{-y} \int_0^\infty \frac{1}{y} e^{-\frac{x}{y}} dx} \\ &= \frac{1}{y} e^{-\frac{x}{y}} \end{aligned}$$

Hence:

$$\begin{aligned} P\{X > 1|Y + y\} &= \int_1^\infty \frac{1}{y} e^{-\frac{x}{y}} dx \\ &= -e^{-\frac{x}{y}}|_1^\infty \\ &= e^{-\frac{1}{y}} \end{aligned}$$

Solution 8

a) Here we take $N = 20$ and first note that the mean of $\sum_{k=1}^N X_k$ -s Nm and $var(\sum_{k=1}^N X_k) = N\sigma^2$.

Therefore in the first part we need to calculate:

$$P(|\frac{1}{20}\sum_{i=1}^{20} X_i - m| < 0.1) = 1 - P(|\frac{1}{20}\sum_{i=1}^{20} X_i - m| > 0.1)$$

Using Chebychev's inequality with $\varepsilon = 0.1$ we have:

$$P(|\frac{1}{20}\sum_{i=1}^{20} X_i - m| > 0.1) = P(|\sum_{i=1}^{20} X_i - 20m| > 2) \leq \frac{20 \times 5}{4} = 25$$

which is meaningless. So Chebychev's inequality is not useful and thus we cannot estimate probability of estimating the mean to within 0.1 using 20 samples via Chebychev's inequality.

b)

In this part we want to find N so that the probability of estimating the mean to an accuracy of 0.1 need to find the smallest N such that:

$$P(|\frac{1}{N}\sum_{k=1}^{N} X_k - m| \leq 0.1) \geq 0.99$$

Thus we need to calculate:

$$P(|\frac{1}{N}\sum_{k=1}^{N} X_k - m| > 0.1) \leq 0.01$$

. Using Chebychev's inequality we estimate the probability to be $\leq \frac{5}{N0.01} \leq 0.01$ or $N \geq \frac{5}{0.0001} =$ 50000.