ECE316- Probability and Random Processes Winter 2011
Problem Set # 4

R. Mazumdar **Week 6**

IT IS IN YOUR INTEREST TO DO THE PROBLEMS YOURSELVES

1. The following problems deal with independence and conditioning for discrete r.v's:
   (a) Yoshi's database of friends contains n entries, but due to a software glitch, the addresses correspond to the names in a totally random fashion. Yoshi sends a holiday card to each of his friends using the corrupted addresses. What is the probability that at least one of his friends will get the correct card? Show that the probability converges to $1 - e^{-1}$ as $n \to \infty$. Hint: Use conditioning and the inclusion-exclusion principle.
   (b) (Random splitting of Poisson random variables) A transmitter sends out either a 1 with probability p or a 0 with probability $1 - p$. Successive transmissions are independent of each other. If the number of transmissions within a given time interval are Poisson with parameter λ, show that the number of 1's transmitted is Poisson with parameter $p\lambda$ and the number of 0's is Poisson with parameter $(1-p)\lambda$.
   (c) Let $X, Y,$ *and* Z be discrete r.v's. Show that the following relationship hold between the conditional pmf's:
   $$p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x,y)$$
   Note here x, y, z denote the generic discrete values that are taken by X, Y and Z respectively.
2. Let X be an uniformly distributed on the interval $[c, d]$. Find the probability density of
   (a) $Y = aX + b$ where a and b are arbitrary constants.
   (b) $Y = \frac{1}{X}$.
   (c) $Y = X^2$
   (d) $Y = \sqrt{X}$
3. Let X be a $N(m, \sigma^2)$ random variable. Find the probability density of $Y = e^X$.
4. Let $X(\omega)$ and $Y(\omega)$ be jointly distributed, non-negative random variables. Show that:
   $$\mathbf{P}(X + Y > z) = \mathbf{P}(X > z) + \mathbf{P}(X + Y > z \geq X)$$
   and
   $$\int_0^\infty \mathbf{P}(X + Y > z \geq X)dz = E[Y]$$

5. Let X be a continuous random variable that takes values in $[0, C]$. Show that:

$$var(X) \leq \frac{C^2}{4}$$

Hint: Argue that $E[X^2] \leq CE[X]$.

6. Let X and Y be two jointly distributed real valued r.v's with joint probability density function $f_{X,Y}(x, y)$. Define $R = \frac{X}{Y}$.

   Find the probability density function of R in terms of $p_X(.), f_Y(.),$ *and* $f_{X,Y}(x, y)$ Specialize your answer to the case when X and Y are independent.

7. Suppose that the joint density of X and Y is given by:

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{e^{-\frac{x}{y}} e^{-y}}{y} \quad 0 < x < \infty, 0 < y < \infty \\ &= 0 \quad otherwise \end{aligned}$$

   Find $P\{X > 1 | Y = y\}$.

8. Let $\{X_i\}$ be independent, identically distributed (i.i.d). r.v's with unknown mean m and variance 5 that is known exactly. We estimate the mean of the random variables by $\hat{m} = \frac{1}{N} \sum_{i=1}^{N} X_i$.

   Use Chebychev's inequality to answer the following questions:

   (a) What is the probability that if I take$N = 20$ samples to estimate the mean I will be within an absolute error of 10^{-1}?

   (b) I need to estimate the mean to an accuracy of 1%. What is the minimum number of samples that I will need to take if I need to be confident of my answer with at least 99% certainty?