ECE316- Probability and Random Processes Winter 2011
Problem Set # 3

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The following are problems on independence, distributions and random variables. The corresponding material can be found in Chapters 3 and 4 of your text.

1. Two events A and B are conditionally independent given an event C if

$$P(AB|C) = P(A|C)P(B|C)$$

Let $A_1, A_2, \cdots, A_n$ be a collection of events such that $P(A_k|A_1A_2\cdots A_l) = P(A_k|A_l)$ where $l < k$. Show that the multiplication law can be written as:

$$P(A_1A_2\cdots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2)\cdots P(A_{i+1}|A_i)\cdots P(A_n|A_{n-1})$$

Furthermore show that $P(A_kA_l|A_j) = P(A_k|A_j)P(A_l|A_j)$ where $k < j < l$ or A_k and A_l are conditionally independent given A_j.

Such events are said to be Markov dependent.

2. Let A and B be independent events. Show at A and B^c are independent. Show that if $P(A|B) = P(A|B^c)$ then A and B are independent.

3. A random variable $X(\omega)$ takes non-negative values and has the probability distribution function given by:

$$\begin{aligned} F(x) = \mathbf{Pr}\{X(\omega) \leq x\} &= 1 - e^{-2x} \quad ; x \geq 0 \\ &= 0 \ \ otherwise \end{aligned}$$

a) Calculate the following: $\mathbf{Pr}.\{X(\omega) \leq 1\}$, $\mathbf{Pr}\{X(\omega) > 2\}$ and $\mathbf{Pr}\{X(\omega) = 3\}$.

b) Find the probability density function $p_X(x)$ of $X(\omega)$.

c) Let $Y(\omega)$ be a r.v. obtained from $X(\omega)$ as follows:

$$\begin{aligned} Y(\omega) &= 0 \ \ if \ X(\omega) \leq 2 \\ &= 1 \ if \ X(\omega) > 2 \end{aligned}$$

Find the probability $\mathbf{Pr}.(Y(\omega) = 0)$.

4. Let $T(\omega)$ be a geometrically distributed r.v.: i.e,

$$\Pr(T = k) = pq^{k-1}, \ p = 1 - q$$

Show that the geometric distribution has no memory, i.e.:

$$\Pr(T > n_0 + k | T > n_0) = \Pr(T > k), \quad k \geq 1$$

5. Let $X(\omega)$ be a Poisson r.v. with parameter λ.

   Show that $E[X^2(\omega)] = \lambda + \lambda^2$. Hence find $var(X)$.

6. Let $X(\omega)$ be a non-negative integer-valued r.v.. Show that:

$$E[X^2] = 2\sum_{k=1}^{\infty} kP(X \geq k) - E[X]$$

7. Let $X(\omega)$ be a real valued r.v. with distribution function $F(x)$.

   In class we showed that If $X(\omega) \geq 0$, we could write:

$$\mathbf{E}[X(\omega)] = \int_0^{\infty} (1 - F(x))dx$$

   Now, if $-\infty < X(\omega) < \infty$ and the distribution is continuous:

$$\mathbf{E}[X(\omega)] = \int_0^{\infty} (1 - F(x))dx - \int_{-\infty}^{0} F(x)dx$$

8. Let X be a continuous r.v. with density function $p_X(x) = C(x - x^2) \ \ x \in [a, b]$.

   (a) What are the possible values of a and b ?

   (b) What is C?

9. Let $X \geq 0$ be a real-valued non-negative random variable with $0 < \mathbf{E}[X^2] < \infty$.

   Show that

$$\sum_{k=1}^{\infty} \mathbf{P}(X \geq k) \leq \mathbf{E}[X] \leq 1 + \sum_{k=1}^{\infty} \mathbf{P}(X \geq k)$$

   Note here $X(\omega)$ is continuous while if X is discrete then

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k)$$

10. Let $X(\omega)$ be a geometric r.v. with parameter p.

    Show that

$$E[\frac{1}{X(\omega)}] = -\frac{p\log p}{1-p}$$

    Hint: $\int_p^1 \frac{1}{x} dx = -\log p$

11. Let $X(\omega)$ be a Bernoulli r.v. with $P(X=1) = p = 1 - P(X=0)$.

    a) Find $var(X)$.

    b) Let Y= (a-b)X + b. Find the distribution of Y and the mean and variance of Y.