ECE316- Probability and Random Processes Winter 2011
Problem Set # 5

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The following problems are related to the material from Chapters 6 and 7 of the text.

1. Let X and Y be two jointly distributed, real valued r.v's with $E[X^2] < \infty$ and $E[Y^2] < \infty$. Suppose we want to approximate X in the mean squared sense by a linear function f Y i.e. by $aY + b$. Show that the optimal choice for a and b are:

$$\begin{aligned} a &= \frac{cov(X,Y)}{var(Y)} \\ b &= m_X - a.m_Y \end{aligned}$$

where $E[X] = m_X$, $E[Y] = m_Y$ and $cov(X,Y) = E[(X - m_X)(Y - m_Y)]$ and $var(Y) = cov(Y,Y) = E[(Y - m_Y)^2]$.

2. Let X be $N(0,1)$ and for fixed x define:

$$\begin{aligned} Z &= X \;\; if \;\; X > x \\ &= 0 \;\; otherwise \end{aligned}$$

In otherwords: $Z = X\mathbf{1}_{[X>x]}$.

Show that:

$$\mathbf{E}[Z] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

3. Show that jointly distributed continuous distributed random variables are independent if and only if their joint density can be expressed as:

$$f_{X,Y}(x,y) = h(x)g(y), \quad -\infty < x < \infty; \quad -\infty < y < \infty$$

Show that for example, two random variables X and Y with joint density, :

$$f_{X,Y}(x,y) = Cxy, \quad 0 < x < 1; \; 0 < y < 1, 0 < x + y < 1$$

cannot be independent.

4. Let Z, Y and Z be independent, identically distributed random variables having uniform distributions over $[0,1]$.

   (a) Find $\mathbf{E}[\max(X,Y,Z))]$

   (b) Find $\mathbf{E}[\min(X,Y,Z)]$

5. The joint density of X and Y is given by:

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-(y+\frac{x}{y})}, \quad x,y > 0$$

Find $\mathbf{E}[X]$, $\mathbf{E}[Y]$ and show that $cov(X,Y) = 1$.

6. If X_1, X_2, X_3, X_4 are pariwise uncorrelated random variables each having mean 0 and variance 1, compute the correlations of:

   (a) $X_1 + X_2$ and $X_2 + X_3$;

   (b) $X_1 + X_2$ and $X_3 + X_4$.

7. The joint density of X and Y is given by:

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-\frac{x}{y}} e^{-y}, \quad x,y > 0$$

Compute $\mathbf{E}[X^2 | Y = y]$.

8. Show that $\mathbf{E}[(X-a)^2]$ is minimized when $a = \mathbf{E}[X]$.

   Now suppose X has a density $f_X(x)$. Then show that $\mathbf{E}(|X-a|)$ is minimized when a is equal to the median of $f_X(.)$. Note the median is defined at the point where half the probability is below that point (the 50th. percentile).

   This shows that one needs to take care of the function being minimized.

9. Show that if X and Y are identically distributed but not necessarily independent then:

$$cov(X+Y, X-Y) = 0$$

10. Prove the Cauchy-Schwarz inequality.

$$(\mathbf{E}[XY])^2 \leq \mathbf{E}[X^2]\mathbf{E}[Y^2]$$

11. Show that if $Y = a + bX$ then:

$$\begin{aligned} \rho(X,Y) &= +1 \quad if \;\; b > 0 \\ &= -1 \quad if \;\; b < 0 \end{aligned}$$

where

$$\rho(X,Y) = \frac{cov(XY)}{\sqrt{var(X)}\sqrt{var(Y)}}$$

is known as the correlation coefficient..