ECE 316-Problem Set 2

Problem 1

a) P(A)=0.1, P(B)=0.9 (typo in PSET), P(A∪B)=0.91

$$\begin{aligned} &P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ \implies &P(A \cap B) = 0.09 = P(A)P(B) \end{aligned}$$

Therefore A and B are independent

b) P(A)=0.4, P(B)=0.6,

$$\begin{aligned} P(A \cup B) &= 0.76 \\ &= P(A) + P(B) - P(A \cap B) \\ \implies P(A \cap B) &= 0.24 \\ &= P(A)P(B). \end{aligned}$$

Therefore A and B are independent.

c) P(A)=0.5, P(B)=0.7,

$$\begin{aligned} P(A \cup B) &= 0.73 \\ &= P(A) + P(B) - P(A \cap B) \\ \implies P(A \cap B) &= 0.47 \neq P(A)P(B). \end{aligned}$$

Therefore A and B are not independent.

Problem 2:

a) For any events A and B, $P(A|A \cup B) \geq P(A|B)$

$$\begin{aligned} &\text{We know that} A \cap B \subseteq A \\ &\text{Hence } P(A) \geq P(A \cap B) \\ &\implies P(A)(P(B) - P(A \cap B)) \geq P(A \cap B)(P(B) - P(A \cap B)) \\ &(\text{Since} P(B) - P(A \cap B) \geq 0, \text{the inequality doesn't change}) \\ &\implies P(A)P(B) \geq P(A \cap B)[P(A) + P(B) - P(A \cap B)] \\ &\implies P(A)P(B) \geq P(A \cap B)[P(A \cup B)] \\ &\implies \frac{P(A)}{P(A \cup B)} \geq \frac{P(A \cap B)}{P(B)} \\ &\implies \frac{P(A \cap (A \cup B))}{P(A \cup B)} \geq \frac{P(A \cap B)}{P(B)} \\ &\implies P(A|A \cup B) \geq P(A|B) \end{aligned}$$

Will your answer be true if we consider the situation

$$P(A|B \cup C) \geq P(A|B)$$

This is not true for arbitrary events. If $D = A^c$, and if we consider that the above relation is true, we can write

$$1 - P(A|B \cup C) \leq 1 - P(A|B)$$

or

$$P(D|B \cup C) \leq P(D|B).$$

So the relation doesn't hold for arbitrary events.

b) Define $P_C(A) = P(A|C)$ To show that it defines a probability, we need to show that it satisfies the axioms of probability.
i) $0 \leq P_C(A) \leq 1$

$$P(A \cap C) \text{ and } P(C), \text{both are } \geq 0. \text{ Hence } P_C(A) = \frac{P(A \cap C)}{P(C)} \geq 0$$

$$\text{Also since } P(A \cap C) \leq P(C) \implies P(A|C) = \frac{P(A \cap C)}{P(C)} \leq 1$$

ii) $P_C(S) = 1$

$$P_C(S) = P(S|C) = \frac{P(S \cap C)}{P(C)} = \frac{P(C)}{P(C)} = 1$$

iii) If $E_i, i = 1, 2, ...$ are mutually exclusive events, then

$$P\left(\left(\bigcup_i E_i\right)|C\right) = \sum P(E_i|C)$$

Proof:

$$\begin{aligned} P(\cup E_i|C) &= \frac{P((\cup E_i) \cap C)}{P(C)} \\ &= \frac{P(\cup(E_i \cap C))}{P(C)} \\ &= \sum \frac{P(E_i C)}{P(C)} \qquad (\text{If } E_i E_j = \phi, \ then \ E_i C E_j C = \phi) \\ &= \sum P(E_i|C) \end{aligned}$$

c) $P_C(A|D) = P(A|C \cap D)$

$$P_C(A|D) = \frac{P_C(A \cap D)}{P_C(D)} = \frac{\frac{P(A \cap D \cap C)}{P(C)}}{\frac{P(D \cap C)}{P(C)}} = P(A|D \cap C)$$

Problem 3

P(Luggage is not missing in Tokyo)=P(Luggage is not misplaced in Toronto $\cap$ Luggage is not misplaced in Chicago $\cap$ Luggage is not misplaced in LA)=$(1-p)^3$, because of independence.

Therefore P(Luggage is missing in Tokyo)= $1-(1-p)^3$
P(Luggage was misplaced in Toronto|Luggage is missing in Tokyo)= $\frac{p}{1-(1-p)^3}$
P(Luggage was misplaced in Chicago|Luggage is missing in Tokyo)= $\frac{(1-p)p}{1-(1-p)^3}$
P(Luggage was misplaced in LA|Luggage is missing in Tokyo)= $\frac{(1-p)^2p}{1-(1-p)^3}$

Problem 4

Let W_1 be the event that the first watch is OK and W_2 be the event that the second watch is OK. We need to find out $P(W_2|W_1)$

$$\begin{aligned}
P(W_2|W_1) &= \frac{P(W_2 \cap W_1)}{P(W_1)} \\
P(W_2 \cap W_1) &= P(W_2 \cap W_1|A)P(A) + P(W_2 \cap W_1|B)P(B) \\
&= \left(\frac{99}{100}\right)^2 \cdot \frac{1}{2} + \left(\frac{199}{100}\right)^2 \cdot \frac{1}{2} \\
\text{and} P(W_1) &= P(W_1|A)P(A) + P(W_1|B)P(B) = \left(\frac{99}{100}\right) \cdot \frac{1}{2} + \left(\frac{199}{100}\right) \cdot \frac{1}{2} \\
P(W_2|W_1) &= \frac{\left(\frac{99}{100}\right)^2 + \left(\frac{199}{100}\right)^2}{\left(\frac{99}{100}\right) + \left(\frac{199}{100}\right)}
\end{aligned}$$

The watches are clearly not independent as $P(W_1 = 1, W_2 = 1) \neq P(W_1 = 1)P(W_2 = 1)$. They are however conditionally independent on the factory they come from.

Problem 5

Let E be the event that the exposed face is red and H be the event that the hidden face is red. Then

$$\begin{aligned}
P(H|E) &= \frac{P(H \cap E)}{P(E)} \\
&= \frac{P(\text{Both sides are red})}{P(\text{At least one side is red})} = \frac{\frac{1}{3}}{\frac{3}{6}} = \frac{2}{3}
\end{aligned}$$

Problem 6

A- Al is innocent
B- Rob testifies that Al is innocent
C- Tom testifies that Al is innocent
X- Witnesses give conflicting testimony
Y- Rob commits perjury (gives false testimony)
Z- Tom commits perjury (gives false testimony)
R- Rob tells the truth
T- Tom tells the truth

a)

$$\begin{aligned}
P(X) &= P(X|A)P(A) + P(X|A^c)P(A^c) \\
P(X|A) &= P(RT^c|A) + P(R^cT|A) = 1 * 0.3 + 0 * 0.7 = 0.3 \\
P(X|A^c) &= P(RT^c|A^c) + P(R^cT|A^c) = 0.8 * 0 + 0.2 * 1 = 0.2 \\
P(X) &= 0.3 * 0.2 + 0.2 * 0.8 = 0.22
\end{aligned}$$

b)

$$\begin{aligned}
P(\text{Rob commits perjury}) &= P(Y) \\
&= P(Y|A)P(A) + P(Y|A^c)P(A^c) \\
&= 0 * 0.2 + 0.2 * 0.8 = 0.16 \\
P(\text{Tom commits perjury}) &= P(Z) \\
&= P(Z|A)P(A) + P(Z|A^c)P(A^c) \\
&= 0.3 * 0.2 + 0 * 0.8 = 0.06
\end{aligned}$$

Rob is more likely to commit perjury

c)

$$\begin{aligned}
P(A|X) &= \frac{P(A \cap X)}{P(X)} \\
&= \frac{P(X|A)P(A)}{P(X)} \\
&= 0.3 * 0.2/0.22 = 3/11
\end{aligned}$$

d)

$$\begin{aligned}
P(YZ) &= P(YZ|A)P(A) + P(YZ|A^c)P(A^c) \\
&= P(Y|A)P(Z|A)P(A) + P(Y|A^c)P(Z|A^c)P(A^c) = 0 + 0 = 0
\end{aligned}$$

Therefore $P(YZ) \neq P(Y)P(Z)$

Problem 7

Let M denote the event that the patient is ill, + denote the event that the test is positive, - denote the event that the test is negative. We need to find the probability that a person who tests negative has the disease=P(M|-) (Note that the question says that “you need to compute P(-|M)” which is wrong)

$$
\begin{aligned}
P(M|-) &= \frac{P(M\cap -)}{P(-)} \\
&= \frac{P(-|M)P(M)}{P(M)P(-|M)+P(M^c)P(-|M^c)} \\
P(M) &= 1/1000 = 0.001 \\
P(M^c) &= 0.999 \\
P(-|M) &= 1/100 = 0.01 \\
P(-|M^c) &= 0.98 \\
\text{Therefore}\quad P(M|-) &= 1.0214 * 10^-5
\end{aligned}
$$

Problem 8

Let L = MAX(X;Y) and S = MIN(X;Y)

We need to compute $P(L \geq 3/4|S \leq 1/3)$

Let us find the distribution of S first:

$$
\begin{aligned}
P(S \leq s) &= P(X \leq s \cup Y \leq s) \\
&= P(X \leq s) + P(Y \leq s) - P(X \leq s \cap Y \leq s) \\
\text{Therefore}\quad P(S \leq 1/3) &= 1/3 + 1/3 - (1/3)^2 = 5/9 \\
P(L \geq 3/4 \cap S \leq 1/3) &= P(X \geq 3/4 \cap Y \leq 1/3) + P(Y \geq 3/4 \cap X \leq 1/3) \\
&= 1/4 * 1/3 + 1/4 * 1/3 = 1/6 \\
\text{So}\quad P(L \geq 3/4|S \leq 1/3) &= \frac{1/6}{5/9} = 3/10
\end{aligned}
$$