ECE316- Probability and Random Variables Winter 2007
In-term Exam 1

Time: 1 Hour 15 minutes

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INSTRUCTIONS

- Enter your name, student ID number, e-mail address and sign in the space provided at the bottom of this page.
- The exam is **1 hour and 15 mins** long.
- DO ALL QUESTIONS. The weightage of each question is indicated.
- Some useful results and formulae are given on the last page.
- You are **not** allowed the use of calculators or crib sheets.
- Unless specifically mentioned to the contrary show all relevant work in the space provided. You may use the extra blank pages and the back of each page if necessary.

Name:

Student ID #:

SOLUTION

E-mail :

Signature:

PART I

(20 %)

These are short problems. For problems involving TRUE or FALSE answers it is not necessary to justify your answers. Only correct answers count- no partial credit.

1. (4pts) Let A and B be two events defined on $(\Omega, \mathcal{F}, P)$: Then:

   a) $\Pr(A) + \Pr(B) \leq 1$. Consider 2 events with $P(A) > 0.5$ and $P(B) > 0.5$

   Answer: True False Then $P(A) + P(B) > 1$

   b) $\Pr(A \cup B) \geq \max\{\Pr(A), \Pr(B)\}$

   Answer: True False

   Since both $A \subset A \cup B$ and $B \subset A \cup B$

2. (6pts) Suppose A, B, C are three events defined on the same probability space.

   a) The events A and B are independent if $A \cap (B^c) = \emptyset$ If $A \cap B^c = \phi$ then

   Answer: True False And B^c are dependent and so are $A + B$

   b) If A and B are independent, and B and C are independent. Then A and C are independent.

   This info tells you nothing about A and C

   Answer: True False

   c) Suppose (A, B), (B, C) and (C, A) are pairs of independent events, then A, B, C are independent.

   Answer: True False

   Pairwise independence is not enough.
   See class notes. - example

3. (4 pts) Let A and B be two events with $\Pr(A) > 0$ and $\Pr(B) > 0$.

   Show that $\Pr(A|B) \geq \Pr(A)$ then $\Pr(B|A) \geq \Pr(B)$.

   $$P(A \cap B) = P(A|B) \cdot P(B) = P(B|A)\, P(A)$$

   So. if $P(A|B) \geq P(A) \Rightarrow P(A \cap B) \geq P(A) P(B)$

   Hence $P(B|A) = \dfrac{P(A \cap B)}{P(A)} \geq \dfrac{P(A) P(B)}{P(A)} \geq P(B)$

4. (6pts) An event B is said to suggest event A if $\Pr(A|B) \geq \Pr(A)$ and doe not suggest it if $\Pr(A|B) < \Pr(A)$. Show that if $0 < \Pr(B) < 1$ then the following is true:

B suggests A if and only if B^c does not suggest A.

Law of total Prob.

$P(A) = P(A|B)\,P(B) + P(A|B^c)\,P(B^c)$ (i)

On the other hand

$P(A) = P(A)\left[P(B) + P(B^c)\right]$ — (ii)

So. (i) − (ii) = 0

$P(B)\left[P(A|B) - P(A)\right] = P(B^c)\left[P(A) - P(A|B^c)\right]$

So clearly if $P(A|B) \geq P(A)$ (B suggest A)

$\Longleftrightarrow$ $P(A) \leq P(A|B^c)$ or B^c does not suggest A.

PART II

For the problems in this section show all calculations in the space provided or at the back of the page. Justify all your answers.

Problem 1: (16%)

Two numbers X and Y are selected at random between -1 and 1. Let the events A, B, C and D be defined as follows: $A = \{X > 0\}, B = \{Y > 0\}, C = \{X < -1/2 \text{ and } Y > 1/2\}$ and $D = \{X < 0 \text{ and } Y < 0\} \cup \{X > 0 \text{ and } Y > 0\}$.

a) Are A and B independent? Why?

X and Y are independent (they are selected at random)
so A and B are independent

b) Are A and C independent? Why?

$A \perp\!\!\!\perp C$: No.

$A \cap C = \{x > 0\} \cap \{\{x < -\frac{1}{2}\} \cap \{y > 1/2\}\}$
$= \phi$

so A and C cannot be independent

c) Are A, B and C independent? Why?

$(A \perp\!\!\!\perp B \perp\!\!\!\perp C)$? No. Since $A \cap B \cap C = \Phi$

as in part b)

d) Are A, B and D independent? Why? Find $\Pr(A \cap B \cap D)$.

$$A \cap B \cap D = \{X>0\} \cap \{Y>0\} \cap \left\{ \{X<0\} \cap \{Y<0\} \cup \{X>0\} \cap \{Y>0\} \right\}$$

$$= \{X>0\} \cap \{Y>0\} = A \cap B$$

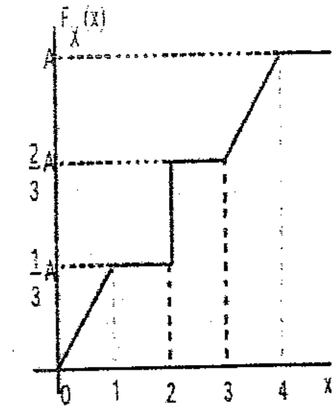
So A, B and D would be independent iff. $P(D) = 1$ which is clearly not the case. So. A, B and D are not independent

$$P(A \cap B \cap D) = P\{(X>0) \cap \{Y>0\}\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

(since X and Y are uniform on $[-1, 1]$)

Problem 3: (24%)

A random variable X has the following cumulative probability distribution function (c.d.f) $F(x)$ shown:

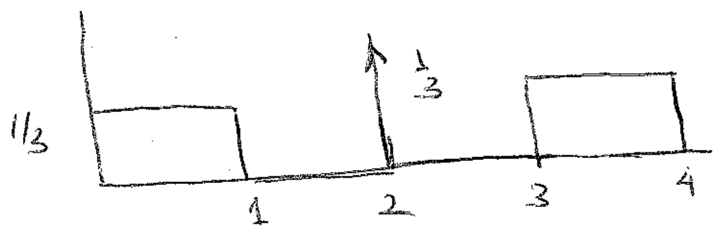


1. What is the value of A? Why?

$A = 1$ since $F(x) \rightarrow 1$ as $x \rightarrow \infty$

2. Sketch the probability density function.

$P_X(x) = F'(x)$



3. Find $\mathbf{E}[X]$.

$$E[X] = \frac{1}{3}\int_0^1 x\,dx + \frac{2}{3} + \frac{1}{3}\int_3^4 x\,dx$$

$$= \frac{1}{6} + \frac{2}{3} + \frac{1}{3}\left[\frac{16}{2} - \frac{9}{2}\right]$$

$$= \frac{1}{6} + \frac{4}{6} + \frac{7}{6} = \frac{12}{6} = 2$$

4. What is the probability that $X < 2$?

$P\{X < 2\} = \frac{1}{3}$ (Right continuous so value to the left of 2)

5. Find the variance of X.

Variance of $X = E[X^2] - E[X]^2$

$$E[X^2] = \frac{1}{3} \cdot \frac{1}{3} + \frac{4}{3} + \frac{1}{3}\int_3^4 x^2\,dx$$

$$= \frac{1}{9} + \frac{4}{3} + \frac{64-27}{9}$$

$$= \frac{1+12+37}{9} = \frac{50}{9}$$

$$\text{Var } X = \frac{50}{9} - 4 = \frac{14}{9}$$

Problem 4: (16%)

Let X be a r.v. with probability mass function (p.m.f) given by:

$$p_k = \Pr(X = k) = \frac{k^2}{C}, \quad k = 0, \pm 1, \pm 2, \pm 3$$
$$= 0, \; otherwise$$

1. Find C and $\mathbf{E}[X]$.

$\sum P_k = 1 = \frac{2}{c}\left[1 + 4 + 9\right] \Rightarrow c = 28$

$E[X] = 0$ by symmetry

2. Find the p.m.f. of the random variable Z defined as:

$$Z = \Big(X - \mathbf{E}[X]\Big)^2$$

Now $Z = X^2$ so the pmf of Z

Z takes values $1, 4, 9$

$P\{Z=0\} = 0$

$P\{Z = 1\} = \frac{1}{28} + \frac{1}{28} = \frac{1}{14}$

$P\{Z=4\} = 2 \times \frac{4}{28} = \frac{8}{28} = \frac{2}{7}$

$P\{Z = 9\} = 2 \times \frac{9}{28} = \frac{18}{28} = \frac{9}{14}$

3. Find the variance of X.

By defn. of Z

$$\text{Var } X = E[Z]$$

$$= \frac{1}{14} + \frac{8}{7} + \frac{81}{14}$$

$$= \frac{1+16+81}{14} = \frac{98}{14} = 7$$

Problem 5: (24%)

Let X and Y be independent geometric random variables with parameter p i.e.:

$$\Pr(X = n) = \Pr(Y = n) = (1-p)^{n-1}p, \quad n \geq 1$$

1. Find the moment generating function of X and hence $\mathbf{E}[X]$.

$$\Phi(z) = \sum^{\infty} z^k (1-p)^{k-1} p$$

$$= pz \sum_1^{\infty} \left[z(1-p)\right]^{k-1}$$

$$= pz \sum_0^{\infty} \left[z(1-p)\right]^{k}$$

$$= \frac{pz}{1 - z(1-p)}.$$

$$E[X] = \frac{d}{dz}\Phi(z)\Big|_{z=1} = \frac{p}{1-z(1-p)}\Big|_{z=1}$$

$$+ \frac{pz(1-p)}{(1-z(1-p))^2}\Big|_{z=1}$$

$$= \frac{p}{p} + \frac{p(1-p)}{p^2}$$

$$= 1 + \frac{(1-p)p}{p^2}$$

$$= \frac{p}{p^2} = \frac{1}{p}.$$

(See Class notes) HW-3 also.

2. Show that

$$\Pr(X > n + m | X > m) = \Pr(X > n)$$

$$P\{X > n+m \mid X > m\} = \frac{P\{(X > n+m) \cap (X > m)\}}{P\{X > m\}} = \frac{P\{X > n+m\}}{P\{X > m\}}$$

$$P\{X > k\} = p \sum_{k+1}^{\infty} (1-p)^{n-1} = p \sum_{k}^{\infty} (1-p)^{n} = (1-p)^{K}$$

$$\text{So } P\{X > n+m \mid X > m\} = \frac{(1-p)^{n+m}}{(1-p)^{m}} = (1-p)^{n} = P\{X > n\}$$

(See HW-3)

3. Find $\Pr(X + Y = n)$

Now

$$P\{X+Y=n\} = \sum_{k=1}^{n-1} P\{X=k, Y=n-k\} = \sum_{k=1}^{n-1} P_k P_{n-k}$$

$$= \sum_{k=1}^{n-1} (1-p)^{k-1} p \, (1-p)^{n-k-1} p$$

(see lecture notes).

$$= p^2 (1-p)^{n-2} \sum_{k=1}^{n-1} 1$$

$$= (n-1)\, p^2 (1-p)^{n-2}$$

4. Show that

$$\Pr(X = k | X + Y = n + 1) = \frac{1}{n}, \qquad k = 1, 2, \ldots, n$$

$$P\{X = k \mid X + Y = n+1\} = \frac{P\{X = k, X + Y = n+1\}}{P\{X + Y = n+1\}}$$

$$= \frac{P\{X = k, Y = n+1-k\}}{P\{X + Y = n+1\}}$$

$$= \frac{p(1-p)^{k-1} \, p(1-p)^{n-k}}{n \, p^2 (1-p)^{n-1}}$$

$$= \frac{1}{n}$$

Note this does not depend on k

so $P\{X = k \mid X + Y = n+1\} = \frac{1}{n}$, $k = 1, 2, \ldots, n$.

Some useful formulae and notes

Abbreviations used: Pr. = Probability, r.v. = Random Variable, c.d.f. = Cumulative Distribution Function (denoted F(x)), p.d.f = Probability density function (for continuous distributions) (usually denoted p(x)), p.m.f= Probability mass function (for discrete distributions) (usually denoted $p_k = \Pr(X = x_k)$.

For right continuous functions $G(x)$ the derivative is defined from the right i.e.:

$$g(x) = \frac{dG(x)}{dx} = \lim_{\Delta - 0} \frac{G(x+\Delta) - g(x)}{\Delta}$$

when it exists.

Some elementary formulae

: If X is a discrete-r.v. then the $k - th$ moment of X:

$$\mathbf{E}[X^k] = \sum_n x_n^k p_n$$

where $p_n = \Pr(X = x_n)$.

If X is continuous then:

$$\mathbf{E}[X^k] = \int_{\Re} x^k dF(x) = \int_{\Re} x^k p(x) dx$$

where $p(x) = \frac{dF(x)}{dx}$is the p.d.f. and $F(x)$ is the c.d.f.

Some other elementary formulae

Given a collection of sets $\{A_i\}$ then De-Morgan's law states:

$$\left(\cup_{k=1}^{n} A_k \right)^c = \cap_{k=1}^{n} A_k^c$$

where $A_k^c = \Omega - A$ (the complement of A_k.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad if \ |r| < 1$$