ECE316-1 Probability Winter 2007
FINAL EXAMINATION

Time: 2 hours 30 mins

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INSTRUCTIONS

- ANSWER ALL QUESTIONS. There are 5 (five) questions in all. You are **not** allowed the use of calculators.
- Some useful results and formulae are given at the end of the paper. These may be detached from the rest of the paper and are not required to be turned in. You are not required to re-prove results proved in class as long as you can state the result precisely.
- Show all relevant work in the space provided. You may use the extra blank pages and the back of each page if necessary.

Name:	
Student ID #:	SAMPLE EXAM WITH SOLUTIONS
E-mail :	
Signature:	

In class we have discussed two alternative schemes for calculating your final grade. Scheme A: 2 Midterms 50% and Final 50%; Scheme B: Best midterm 35% and Final 65%

Write down your choice here:

Problem 1: (20%)

a) When a packet is transmitted on a communications link, the probability that the packet is received with errors is p. Upon receiving a packet the receiver sends back an ACK to the sender. The probability that an ACK is in error is q.

Find the probability that the sender has to re-transmit a packet either because of errors in transmission or because the ACK is in error or both.

Prob(to transmit) = Prob. (error in send or ack or both)

$$= 1 - \text{Prob. (no error)}$$

$$= 1 - \underbrace{(1-p)}_{\text{No error in transmit.}}\underbrace{(1-q)}_{\text{No error in ACK}} = p + q - pq$$

b) Let X be a Binomial B(n,p) r.v. and Y be a Binomial B(m,p) r.v. that is independent of X. Show that $Z = X + Y$ also has a Binomial distribution. What are the parameters of the distribution?

$$P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P\{Y=k\} = \binom{m}{k} p^k (1-p)^{m-k}$$

Hence

$$\begin{aligned}
P\{Z=k\} &= \sum_{j=0}^{k} P\{X=j, Y=k-j\} \\
&= \sum_{j=0}^{k} P\{X=j\} P\{Y=k-j\} \\
&= \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j} \binom{m}{k-j} p^{k-j} (1-p)^{m-(k-j)} \\
&= p^k (1-p)^{n+m-k} \sum_{j=0}^{k} \binom{n}{k}\binom{m}{k-j} \\
&= p^k (1-p)^{n+m-k} \binom{n+m}{k}
\end{aligned}$$

Z is $B(n+m, p)$.

c) Let X and Y be two r.v.'s that take values in $\{0,1\}$ i.e. they can be 0 or 1. Suppose the joint p.m.f. is given by $p(0,0) = a, p(0,1) = b, p(1,0) = c$ and $p(1,1) = d$ where $a+b+c+d = 1$.

Show that X and Y are uncorrelated if $d = (b+d)(c+d)$. What is the condition for X and Y to be independent?

$$E[X] = 0 \cdot P\{X=0\} + 1 P\{X=1\} = P\{X=1\}$$

$P\{X=1\} = c+d \qquad P\{Y=1\} = b+d$

$$\mathrm{Cov}(XY) = E[XY] - E[X]E[Y]$$
$$= p(1,1) - (d+c)(b+d)$$
$$= d - (d+c)(b+d)$$

Hence X and Y are uncorrelated if $\mathrm{Cov}(XY) = 0$

$$\Rightarrow \quad d = (b+d)(c+d)$$

For X and Y to be independent we need

$$P\{X=x, Y=y\} = P\{X=x\} P\{Y=y\}$$

where $x \in \{0,1\} \quad y = \{0,1\}$

$P\{X=0\} = a+b \qquad P\{X=1\} = c+d$

$P\{Y=0\} = a+c \qquad P\{Y=1\} = b+d$.

We need the foll. to be satisfied

$$a = (a+b)(a+c)$$
$$b = (a+b)(b+d)$$
$$c = (c+d)(a+c)$$
$$d = (a+c)(b+d).$$
$$a+b+c+d = 1.$$

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Problem 2: (20%)

Let X and Y be independent r.v's with the following distributions:

i) X is $N(0, \sigma^2)$.

ii) Y takes the value 1 with prob. p and -1 with prob q=1-p

Define $W = XY$.

a) Show that W is Gaussian.

To show W is Gaussian we compute its moment generating function and show it is quadratic

$$E[e^{tW}] = E[e^{tX} \mid Y=1]\, p + E[e^{-tX} \mid Y=-1]\,(1-p)$$

$$= e^{-\frac{1}{2}t^2\sigma^2} \cdot p + e^{-\frac{1}{2}t^2\sigma^2}(1-p) = e^{-\frac{1}{2}t^2\sigma^2}$$

Since $E[e^{tX}] = e^{-\frac{1}{2}t^2\sigma^2} = E[e^{-tX}]$

since X is $N(0,\sigma^2)$

so W is Gaussian

b) Find $\mathbf{E}[W]$ and $var(W)$. Are W and X correlated ? Independent ?

$$E[W] = p\,E[X] + (1-p)\,E[-X]$$
$$= 0$$

$$Var(W) = E[W^2] = E[X^2] = \sigma^2$$

$$E[WX] = E[YX^2] = E[X^2]\,E[Y] = 0$$

$\Rightarrow$ W and X are uncorrelated.

They are clearly not independent since $p(w,x) \neq p(x)\,p(w)$.

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Problem 3: (20%)

a) Let $S \in [a,b]$ be a random signal with p.d.f. $p_S(s)$ and N be noise that is a $N(0,1)$ Normally distributed (or Gaussian) r.v. independent of S. Suppose we observe: $Y = S + N$

Show that the conditional expectation of S given Y can be written as:

$$\mathbf{E}[S|Y=y] = \frac{\int_a^b s e^{-\frac{(y-s)^2}{2}} p_S(s) ds}{\int_a^b e^{-\frac{(y-s)^2}{2}} p_S(s) ds}$$

$$E[S|Y=y] = \int_a^b s\, p(s|y)\, ds$$

Let us compute $p(s|y)$

$$p(s|y) = \frac{p(s,y)}{p(y)} = \frac{p(s,y)}{\int_a^b p(y|s)p(s)\,ds}$$

$$= \frac{p(y|s)\,p(s)}{\int_a^b p(y|s)p(s)\,ds}$$

Now $p(y|s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-s)^2}$

since given s $Y \sim N(s,1)$

Hence $p(s|y) = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{1}{2}(y-s)^2} \cdot p_s(s)}{\cancel{\frac{1}{\sqrt{2\pi}}} \int_a^b e^{-\frac{1}{2}(y-s)^2} p_s(s)\, ds}$

From which the answer follows.

b) Now suppose that S is also a Gaussian $N(0, r^2)$ signal independent of N. Show that

$$\mathbf{E}[e^{tS}|Y] = e^{\{t\frac{r^2}{1+r^2}Y + \frac{1}{2}t^2\frac{r^2}{1+r^2}\}}$$

using only the following facts:

- S and Y are jointly gaussian.
- $S - \mathbf{E}[S|Y]$ is uncorrelated with $f(Y)$ where $f(.)$ is any continuous function.
- Using the properties of conditional expectations.

Hint: Write $S = \mathbf{E}[S|Y] + S - \mathbf{E}[S|Y]$

$$E\left[e^{ts}|Y\right] = E\left[e^{t\left(E[s|Y] + s - E[s|Y]\right)}\Big|Y\right]$$

$$= e^{tE[s|Y]} \cdot E\left[e^{t\left(s - E[s|Y]\right)}\Big|Y\right]$$

Now we know $s - E[s|Y]$ is independent of Y

so $E\left[e^{ts}|Y\right] = e^{tE[s|Y]} \cdot E\left[e^{t(s - E[s|Y])}\right]$

$s - E[s|Y]$ is Gaussian since $E[s|Y] = \frac{r^2}{1+r^2}Y$

$s - E[s|Y]$ is therefore $N\left(0, \operatorname{Var}(s - E[s|Y])\right)$

$$\operatorname{Var}\left(s - E[s|Y]\right) = E[s^2] - E\left[s\, E[s|Y]\right]$$

$$= r^2 - \frac{r^2}{1+r^2}r^2$$

$$= \frac{r^2}{1+r^2}$$

Hence

$$E\left[e^{ts}|Y\right] = e^{t\frac{r^2}{1+r^2}Y + \frac{1}{2}t^2\frac{r^2}{r^2+1}}$$

Problem 4:(20%)

a) Let $\{X_i\}_{i=1}^{n}$ be a collection of i.i.d. r.v's with $\mathbf{E}[X_i] = m$ and $var(X_i) = \sigma^2$
Show that:

$$var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}$$

Since the r.v.'s are indep.

$$Var\left(\frac{1}{n}\sum_{1}^{n} X_i\right) = \sum_{1}^{n} Var\left(\frac{X_i}{n}\right)$$

$$= \frac{\sum_{1}^{n} \sigma^2}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

b) Let X and Y be jointly Gaussian r.v. with mean $\mathbf{E}[X] = \mathbf{E}[Y] = m$ and $var(X) = var(Y) = \sigma^2$. The correlation coefficient is $\rho = \frac{Cov(X,Y)}{\sqrt{var(X)var(Y)}}$

Show that $X + Y$ is independent of $X - Y$.

$$E[X+Y] = E[X] + E[Y] = 2m$$

$$E[X-Y] = E[X] - E[Y] = 0$$

Since X and Y are jointly gaussian

$X+Y$ and $X-Y$ are Gaussian.

so it is enough to show

$Cov(X+Y, X-Y) = 0$ since this will

imply independence

$$Cov(X+Y, X-Y) = E[(X+Y)(X-Y)] - E[X+Y] \cdot E[X-Y]$$

$$= E[X^2 - Y^2] = E[X^2] - E[Y^2] = 0$$

so X and Y are independent

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Problem 5: (20%)

a) Suppose that orders at a restaurant are iid (independent and identically distributed) random variables with mean $m = \$8$ and standard deviation $\sigma = \$2$.

i) Using the Central Limit Theorem, estimate the probability that the first 100 customers spend a total of more than \$840.

ii) After how many orders can we be 90% sure that the total spent by all the customers is more than \$1000? Pose the problem. Do not try to solve it.

(i) Let X_i be the order of customer i

$$P\left(\sum_1^{100} X_i > 840 \right)$$

We know $E[X_i] = 8$ $\quad Var(X_i) = 4$

$$P\left\{ \sum_1^{100} X_i > 840 \right\} = P\left\{ \frac{\sum_1^{100} X_i - 100 \times 8}{\sqrt{100} \cdot 2} > \frac{840-800}{2\sqrt{100}} \right\}$$

$$= P\left\{ \frac{\sum_1^{100} X_i - 800}{20} > 2 \right\}$$

$$= 1 - \Phi(2)$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$

$$= 1 - .9773 = .0227$$

(ii) We want to find n s.t

$$P\left\{ \sum_1^{n} X_i > 1000 \right\} \geq 0.90$$

$$= P\left\{ \frac{\sum_1^{n} X_i - 8n}{\sqrt{n} \cdot 2} > \frac{100-8n}{\sqrt{n}\,2} \right\} \geq .90$$

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We want $\dfrac{100 - 8n}{2\sqrt{n}} = -1.285$

b) Show that :

$$\lim_{n\to\infty}\sum_{k=0}^{n} e^{-n}\frac{n^k}{k!} = \frac{1}{2}$$

Hint: Use the fact that if X_i are i.i.d Poisson r.v's with mean 1 and variance 1 then via the CLT

$$\lim_{n\to\infty}\Pr\left(\frac{\sum_{i=1}^{n} X_i - n}{\sqrt{n}} > 0\right) = \frac{1}{2}$$

Let $X_i \sim$ Poisson (1)

Then $\sum_{1}^{n} X_i \sim$ Poisson (n). (sums of indep poisson).

$$P\left\{\sum_{i=1}^{n} X_i \leq n\right\} = \sum_{k=0}^{n} e^{-n}\frac{n^k}{k!}$$

By the CLT

$$P\left\{\sum_{i=1}^{n} X_i \leq n\right\} = P\left\{\frac{\sum_{1}^{n} X_i - E[X_i]\cdot n}{\sqrt{n}\,\sigma} \leq \frac{n-n}{\sqrt{n}\sigma}\right\}$$

$$= P\left\{\frac{\sum_{1}^{n} X_i - n}{\sqrt{n}} \leq 0\right\}$$

$$\longrightarrow \Phi(0) = \frac{1}{2} \quad \text{as } n \to \infty$$

Then

$$\sum_{1}^{n} e^{-n}\frac{n^k}{k!} \longrightarrow \frac{1}{2} \quad \text{as } n\to\infty$$

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Some useful formulae and notes

Abbreviations used: Pr. = Probability, r.v. = Random Variable, c.d.f. = Cumulative Distribution Function (denoted F(x)), p.d.f = Probability density function (for continuous distributions) (usually denoted p(x)), p.m.f= Probability mass function (for discrete distributions) (usually denoted $p_k = \Pr(X = x_k)$.

For right continuous functions $G(x)$ the derivative is defined from the right i.e.:

$$g(x) = \frac{dG(x)}{dx} = \lim_{\Delta \to 0} \frac{G(x+\Delta) - g(x)}{\Delta}$$

when it exists.

Some elementary formulae

: If X is a discrete-r.v. then the $k - th$ moment of X:

$$\mathbf{E}[X^k] = \sum_n x_n^k p_n$$

where $p_n = \Pr(X = x_n)$.

If X is continuous then:

$$\mathbf{E}[X^k] = \int_{\Re} x^k dF(x) = \int_{\Re} x^k p(x) dx$$

where $p(x) = \frac{dF(x)}{dx}$ is the p.d.f. and $F(x)$ is the c.d.f.

A discrete r.v. X is to be Binomial (n,p) often denoted $B(n,p)$ if:

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}. \quad k = 0, 1, \cdots, n$$

The following combinatorial identities hold:

$$\binom{n+m}{k} = \sum_{k=0}^{n} \binom{n}{j}\binom{m}{k-j} = \sum_{j=0}^{m} \binom{m}{j}\binom{n}{k-j}$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}; \quad 1 \le r \le n$$

A discrete r.v. is said to be Poisson with parameter λ if:

$$\Pr(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \cdots$$

The distribution function of a standard Normal (or Gaussian) r.v. is given by:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \quad -\infty < x < \infty$$

Conditional densities and expectation

If X and Y are jointly distributed with joint density $p(x,y)$ then the conditional density of X given $Y=y$ is given by:

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$$

The conditional expectation of X given Y is defined as:

$$\mathbf{E}[X|Y=y] = \int x p_{X|Y}(x|y)dx$$

Property of conditional expectation:

- $\mathbf{E}[X|Y] = g(Y)$ where $g(.)$ is some function.
- If X and Y are independent then $\mathbf{E}[f(X)|Y] = \mathbb{E}[f(X)]$
- $\mathbf{E}[f(Y)h(X)|Y] = f(Y)\mathbf{E}[h(X)|Y]$

If X and Y are jointly Gaussian then:

$$\begin{aligned}
\mathbf{E}[X|Y] &= m_X + \frac{cov(X,Y)}{var(Y)}(Y - m_Y) \\
var(X|Y) &= var(X) - \frac{(cov(X,Y))^2}{var(Y)}
\end{aligned}$$

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Moment generating functions If X is a continuous r.v. then the moment generating function of X is defined as:

$$M_X(t) = \mathbf{E}[e^{tX}] = \int e^{tx} p_X(x)dx$$

If X is discrete with values $\{a_k\}$ then:

$$\phi_X(z) = \mathbf{E}[Z^X] = \sum_k z^{a_k} p_k$$

where $p_k = \Pr(X = a_k)$.

Some common mgf

Continuous distributions

- $N(m,\sigma^2)$; $M_X(t) = e^{\{mt + \frac{\sigma^2}{2}t^2\}}$
- Exponential (λ) ; $M_X(t) = \frac{\lambda}{\lambda - t}$

Discrete-distributions:

- Geometric $P(x=k) = (1-p)^{k-1}p, k \geq 1$; $\phi(z) = \frac{pz}{1-pz}$
- Binomial $B(n,p)$; $\phi(z) = (pz+q)^n$ where $q = 1-p$
- Poisson(λ), $\phi(z) = e^{\lambda(z-1)}$

Inequalities

- (Markov Inequality) $\Pr(f(X) \geq a) \leq \frac{\mathbf{E}[f(X)]}{a}$
- (Chebychev Inequality) $\Pr(|X - \mathbf{E}[X]| \geq a) \leq \frac{var(X)}{a^2}$

WLLN and CLT

(Weak law of large numbers (WLLN))

Let $\{X_n\}$ be a sequence of i.i.d r.v.'s with $\mathbf{E}[|X_1|] < \infty$ and $\mathbf{E}[X_1] = m$. Then

$$\frac{1}{n}\sum_{k=1}^{n} X_k = m$$

in probability.

(Central Limit Theorem)

Let $\{X_i\}_{i=1}^n$ be a sequence of i.i.d. r.v's with $\mathbf{E}[X_i^2] < \infty$. Define $S_n = \sum_{i=1}^n X_i$. Then as $n \to \infty$:

$$\Pr\left(\frac{S_n - nm}{\sigma\sqrt{n}} \leq x\right) \to \Phi(x)$$

where $\mathbf{E}[X_1] = m$ and $var(X_1) = \sigma^2$ and $\Phi(x)$ denotes the standard normal distribution given by:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy$$

Some other elementary formulae

Given a collection of sets $\{A_i\}$ then De-Morgan's law states:

$$\left(\cup_{k=1}^n A_k\right)^c = \cap_{k=1}^n A_k^c$$

where $A_k^c = \Omega - A$ (the complement of A_k.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r} \quad if\ |r| < 1$$

Integration by parts:

$$\int_a^b u dv = uv|_a^b - \int_a^b v du$$

Jacobian formula

Let $\mathbf{X}$ be a $\Re^n$ valued r.v. and let $\mathbf{Y} = f(\mathbf{X})$ be a $\Re^n$ valued r.v. and $f(.)$ be a 1:1 mapping.

Let $x_k = [f^{-1}(\mathbf{y})]_k$ i.e. the kth. component of f^{-1}.

Define the so called Jacobian matrix:

$$J_Y(y) = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial y_n} & \frac{\partial x_2}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}$$

Then:

$$P_Y(y) = P_X(f^{-1}(y))|det\ J_Y(y)|$$

$\Phi(x)$ vs. x , The Normalized Gaussian Distribution Function

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998	0.9998
3.50	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.60	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.70	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000