Problem Set 2
ECE603- Statistical Signal Processing
Winter 2014

Note: The midterm will be held on Friday, Feb. 28, 2014.

These problems are all related likelihood ratios, detection and the Neyman-Pearson criterion.

1. Consider the classical Bayes' detection problem with:

   H_1: The distribution is P_1

   H_2: The distribution is P_2

   Assume that $P_2 << P_1$ and let $L(\omega) = \frac{dP_2}{dP_1}(\omega)$ denote the likelihood ratio or Radon-Nikodym derivative. Assume that the prior distributions corresponding to the hypotheses H_1 and H_2 are π_1 and $\pi_2 = 1 - \pi_1$ respectively.

   Let A_1^* be the optimal decision region to decide for H_1 against H_2. It depends on π_1 and hence let us denote the error $P_E(A_1^*)$ by $P_E(\pi_1)$ where:

   $$P_E(A_1) = \pi_1 + \int_{A_1} (\pi_2 L(\omega) - \pi_1) dP_1(\omega)$$

   Show that $P_E(\pi_1)$ achieves a maximum for some $\pi_1 \in (0,1)$. Show that at this value of π_1, P_E is equal to the error of the second kind and $1 - P_E$ is equal to the error of the first kind.

   Recall, an error is said to be of the first kind if we conclude H_2 is true when actually H_1 is true (False alarm). An error is said to be of the second kind if we conclude H_1 when H_2 is true (mis-detection or a miss).

2. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let Q be another probability measure defined on $(\Omega, \mathcal{F})$. Let $Q << P$ and let $L(\omega)$ denote the Radon-Nikodym derivative.

   Let $X(\omega)$ be a r.v. defined on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$.

   Show that:

   $$E_Q[X|\mathcal{G}] = \frac{E_P[LX|\mathcal{G}]}{E_P[Z|\mathcal{G}]}$$

   Use this result to show that if :

   $$Y = S + N$$

and N is $N(0,1)$ independent if S whose density is denoted by $\mu_S(.)$.

Then:

$$E[S|Y=y] = \frac{\int_{\Re} s e^{-\frac{(y-s)^2}{2}} \mu_s(s) ds}{\int_{\Re} e^{-\frac{(y-s)^2}{2}} \mu_s(s) ds}$$

The above result is called the abstract version of Bayes' Formula.

3. Let N_t be a Poisson point on $(\Omega, \mathcal{F})$. Let P_i correspond to the measure under which N_t has intensity (or rate) λ_i. Let $P_{i,t}$ denote the probability measure $P-i$ restricted to $\sigma(N_t)$.

   Show that:

$$\frac{dP_{2,t}}{dP_1, t} = \left(\frac{\lambda_2}{\lambda_1}\right)^{N_t} e^{(\lambda_2 - \lambda_1)t}$$

4. Let P_θ be a family of absolutely continuous probability measures (w.r.t. a reference probability P_{θ_0}). Let $L(\theta)$ be the likelihood ratio $\frac{dP_\theta}{dP}$. Suppose $\theta \in \theta = \{\theta_1, \theta_2, ..., \theta_M\}$ with priors $P(\theta = \theta_i) = \pi_i$. Formulate a Neyman-Pearson test to test whether $\theta = \theta_0$ or $\theta \in \theta$.