1. (a) $Pr\{X(\omega) \le 1\} = F(1) = 1 - e^{-2}$

$Pr\{X(\omega) > 2\} = 1 - F(2) = 1 - (1 - e^{-4}) = e^{-4}$

$Pr\{X(\omega) = 3\} = 0$ Since $X(\omega)$ is a continuous R.V.

(b) pdf. $p_X(x)$ of $X(\omega)$

$$p_X(x) = \frac{d}{dx}\{F_X(x)\} = \begin{cases} 2e^{-2x}, & x \ge 0 \\ 0 & ; \text{ otherwise} \end{cases}$$

(c) $P_Y(Y(\omega) = 0) = Pr\{X(\omega) \le 2\} = 1 - e^{-4}$

$Pr(Y(\omega) = 1) = Pr\{X(\omega) > 2\} = e^{-4}$

$P_Y(y) = (1 - e^{-4})\delta(y) + e^{-4}\delta(y-1)$

2. (a) $X(\omega) \ge 0$

By the def. of $E(X(\omega))$

$$E(X(\omega)) = \int_0^\infty x\, dF(x)$$

Since $dF_X(x) = -d(1 - F_X(x))$

$$E(X(\omega)) = -\int_0^\infty x\, d(1 - F_X(x))$$

Use Integration by parts

$$= -x(1 - F(x))\Big|_0^\infty + \int_0^\infty (1 - F(x))\, dx$$

Since $E|X| < \infty$

$$\lim_{x \to \infty} x(1 - F(x)) = 0$$

$$= 0 + \int_0^\infty (1 - F(x))\, dx = \int_0^\infty (1 - F(x))\, dx$$

$\square$

2. (b)

$$E(X(\omega)) = \int_{-\infty}^{\infty} x\,dF(x)$$

$$= \underbrace{\int_{-\infty}^{0} x\,dF(x)}_{} + \int_{0}^{\infty} x\,dF(x)$$

↳ use integration by parts then,

$$= xF(x)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} F(x)\,dx + \int_{0}^{1}(1-F(x))\,dx$$

Since if $E|X|<\infty$

$$\lim_{x\to-\infty} xF(x) = 0$$

$$= \int_{0}^{\infty}(1-F(x))\,dx - \int_{-\infty}^{0} F(x)\,dx$$

□

3. Assume $A=\{X+Y>z\}$ $B=\{X>z\}$

Use $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

$$\therefore P(X+Y>z) = \underbrace{P(X+Y>z \mid X>z)}_{=1 \text{ since } Y \text{ is non-negative.}} P(X>z) + P(X+Y>z \mid X\le z)P(X\le$$

$$= P(X>z) + P(X+Y>z\ge X)$$

$$\int_0^{\infty} P(X+Y>z\ge X)\,dz = \int_0^{\infty}\big(P(X+Y>z) - P(X>z)\big)\,dz$$

from the result of 2. (a)

$$= E(X+Y) - E(X) = E(Y(\omega))$$

□

3/4

4. (a) $X(\omega)$ takes only two values $(-1, 1)$

$$P_Z(z) = P(z \mid X=-1)\cdot P(X=-1) + P(z \mid X=1)\,P(X=1)$$

$$\begin{cases} P_{Z|X}(z \mid X=-1) = P_{Y|X}(z+1 \mid X=-1) = P_Y(z+1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z+1)^2}{2\sigma^2}} \\ P_{Z|X}(z \mid X=1) = P_{Y|X}(z-1 \mid X=1) = P_Y(z-1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-1)^2}{2\sigma^2}} \end{cases}$$

$$= \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left\{ e^{-\frac{(z+1)^2}{2\sigma^2}} + e^{-\frac{(z-1)^2}{2\sigma^2}} \right\}$$

(b) Part a.

(c) $E(W) = E(XY) \underset{ind.}{=} E(X)\cdot E(Y) = 0$

$Var(W) = E(X^2Y^2) = E(X^2)E(Y^2) = E(Y^2) = \sigma^2$

Uncorrelated ?

$E(W\cdot Y) = E(X\cdot Y^2) = E(X)\cdot E(Y^2) = 0$

$E(W)\cdot E(Y) = 0$ $\quad\therefore\; E(W\cdot Y) = E(W)E(Y) \Rightarrow$ uncorrelated.

Independence?

$E(W^2Y^2) = E(X^2Y^4) = E(X^2)E(Y^4) = 1\cdot(3\cdot\sigma^4) = 3\sigma^4$

$E(W^2)E(Y^2) = \sigma^2\cdot\sigma^2 = \sigma^4$

$\therefore\; E(W^2Y^2) \neq E(W^2)E(Y^2)$ $\quad\therefore$ W & Y are not independent

5. $X_1(\omega)$, $X_2(\omega)$

$$P(X_1, X_2) = \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}$$

$\rightarrow$ jointly Gaussian with mean 0 & variance 1.

$Y(\omega) = \sqrt{X_1^2(\omega) + X_2^2(\omega)}$

Change the coordinate. $\quad \cos\phi = \frac{X_1}{\sqrt{X_1^2+X_2^2}}$ $\quad \sin\phi = \frac{X_2}{\sqrt{X_1^2+X_2^2}}$

Let $X_1 = Y\cos\phi$, $X_2 = Y\sin\phi$

$$P(Y,\phi) = \left| \begin{matrix} \cos\phi & -Y\sin\phi \\ \sin\phi & Y\cos\phi \end{matrix} \right| \frac{1}{2\pi} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} y e^{-\frac{y^2}{2}} \text{ (for } y \geq 0)$$

5. $p(y,\phi) = \frac{1}{2\pi} \cdot y e^{-\frac{y^2}{2}} = p_Y(y) \cdot p_\phi(\phi)$

$\therefore \; p_Y(y) = y e^{-\frac{y^2}{2}} \; ; \; y \geq 0$

$p_\phi(\phi) = \frac{1}{2\pi} \; ; \; |\phi| \leq \pi$

$\therefore$ Y & ϕ are independent.

$E[X_1 \mid Y] = E(Y\cos\phi \mid Y) = Y E(\cos\phi) = 0$ since $E(\cos\phi) = 0$ $\square$

6. $E(X_1 \mid X_2) = E(Y - \alpha X_2 \mid X_2) = E(Y \mid X_2) - E(\alpha X_2 \mid X_2)$

$= E(Y) - \alpha X_2 = E(X_1 + \alpha X_2) - \alpha X_2$

$= \cancelto{0}{E(X_1)} + \alpha \cancelto{0}{E(X_2)} - \alpha X_2 = -\alpha X_2$ $\square$

7. Without loss of generality let us take $E[X] = 0$

$$\varepsilon - X \leq (\varepsilon - X)\, 1_{(\varepsilon > X)}$$

Hence taking expectations

$$\varepsilon \leq E[(\varepsilon - X)\, 1_{(\varepsilon \geq X)}]$$

So

$$\varepsilon^2 \leq [E(\varepsilon - X)\, 1_{(\varepsilon \geq X)}]^2 \leq E(\varepsilon - X)^2 \, P\{X \leq \varepsilon\}$$

$$\Rightarrow \quad P\{X \leq \varepsilon\} \geq \frac{\varepsilon^2}{\varepsilon^2 + \operatorname{Var} X}$$

So

$$P\{X > \varepsilon\} = 1 - P\{X \leq \varepsilon\} \leq 1 - \frac{\varepsilon^2}{\varepsilon^2 + \operatorname{Var} X} = \frac{\operatorname{Var} X}{\varepsilon^2 + \operatorname{Var} X}$$