ECE 604- PSET 2 Solution

1. 1) By definition $g(z) = \sum_{n=0}^{\infty} p_n z^n$ where $p_n = P(X = n)$. Therefore for $z \in [0, 1]$ we have: $g'(z) = \sum_{n=1}^{\infty} n p_n z^{n-1} \geq 0$ and is therefore non-decreasing and

$$g''(z) = \sum_{n=2}^{\infty} n(n-1) z^{n-2} p_n \geq 0$$

and is therefore convex.

2) For $g'(z)$ to be 0 for some $z \in [0, 1]$ we need $p_n = 0 \; \forall n \geq 1$ or $P(X = 0) = 1$. Similarly for $g''(z) = 0$ we need $p_n = 0 \; \forall \; n \; \geq 2$ or $p_0 + p_1 = 1$. Therefore if $0 < p_0 < 1$ $g'(z) > 0$ for $z \in [0, 1]$ and hence $g(z)$ is strictly increasing and if $p(X > 2) > 0 \leftrightarrow p_0 + p_1 < 1$ then $g''(z) > 0$ or $g(z)$ is strictly convex.

3) Now $E[x] = g'(1)$. Therefore if $g'(1) \leq 1$ we see that the curve $g(z)$ has a slope less than that of z which is 1. Noting that $g(0) = p_0$ and $g(1) = \sum_{n=0}^{\infty} p_n = 1$ we see that the two curves $y = z$ and $y = g(z)$ only intersect at z=1 since $g(z)$ is convex increasing. On the other hand if $E[X] > 1$ then necessarily $g'(1) > 1$. If $p_0 > 0$ then the two curves must intersect at $z = 1$ and some $z \in (0, 1)$. Note if $p_0 = 0$ then the curves intersect at $z = 0$ and $z = 1$ since $g(z)$ is strictly convex.

2. Let $Y_n = \max\{X - 1, X_2, \ldots, X_n\}$ where the $\{X_i\}_{i=1}^{n}$ are i.i.d.. Then

$$\begin{aligned} \mathbb{P}(Y_n \leq x) = F_{Y_n}(x) &= \mathbb{P}\left(\bigcap_{i=1}^{n} (X_i \leq x)\right) \\ &= (F(x))^n \end{aligned}$$

Similarly:

$$\begin{aligned} \mathbb{P}(Z_n > x) = 1 - F_{Z_n}(x) &= \mathbb{P}\left(\bigcap_{i=1}^{n} (X_i > x)\right) \\ &= (1 - F(x))^n \end{aligned}$$

Now let $\psi_n = Y_n - Z_n \geq 0$. There are at least two ways of solving this problem.

First way:

$$\begin{aligned} F_{\psi}(z) = \mathbb{P}(\psi_n \leq z) &= \sum_{i=1}^{n} \int_{0}^{\infty} \mathbb{P}(Y_n - Z_n | Z_n = X_i = x) dF_{X_i}(x) \\ &= \sum_{i=1}^{n} \mathbb{P}(\bigcap_{j=1, j \neq i}^{n} (x \leq X_j \leq x + z)) \, dF(x) \\ &= n \int_{-\infty}^{\infty} (F_X(x+z) - F_X(x))^{n-1} \, dF(x) \\ &= n \int_{-\infty}^{\infty} (F(x+z) - F(x))^{n-1} dF(x) \end{aligned}$$

The second is we integrate out w.r.t. the distribution of Z_n. Note $dF_{Z_n}(x) = n(1 - F(x))^{n-1}dF(x)$.

$$
\begin{aligned}
F_\psi(z) = \mathbb{P}(\psi_n \leq z) &= \mathbb{P}(Y_n - Z_n \leq z) \\
&= \int_{-\infty}^{\infty} \mathbb{P}(Y_n \leq Z_n + z | Z_n = x) dF_{Z_n}(x) \\
&= n \int_{-\infty}^{\infty} \mathbb{P}(Y_n \leq x + z) | Z_n = x)(1 - F(x))^{n-1} dF(x) \\
&= n \int_{-\infty}^{\infty} \mathbb{P}\left(\bigcap_{j=1}^{n-1} (X_j \leq x + z) \mid \bigcap_{j=1}^{n-1} (X_j > x) \right) (1 - F(x)^{n-1} dF(x) \\
&= n \int_{-\infty}^{\infty} \frac{(F(x+z) - F(x))^{n-1}}{(1 - F(x))^{n-1}} (1 - F(x))^{n-1} dF(x) \\
&= n \int_{-\infty}^{\infty} (F(x+z) - F(x))^{n-1} dF(x)
\end{aligned}
$$

where we have used the fact that if the minimum is x then the remaining (n-1) random variables must be larger than x. In other words, given that $Z_n = x$ changes the distribution of the max Y_n since Y_n must be the max of the remaining (n-1) random variables and each of them must be larger than x.

3. This problem is trivial since the random variables are independent and identically distributed. By symmetry: $E[X|X+Y] = E[Y|X+Y]$ and $E[X+Y|X+Y] = X+Y$.

   The second result follows from the fact that $E[X_i|S_n] = E[X_j|S_n]$ and therefore $nE[X_i \mid S_n] = E[S_n|S_n] = S_n$.

4. $X \sim N(0, \sigma^2)$ and $Y \in \{-1, 1\}$ *with* $\mathbb{P}(Y = 1) = p = 1 - \mathbb{P}(Y + -1)$.

   (a) $Z = X + Y$. Then the characteristic function of Z is given by: $C_Z(h) = E[e^{jhZ}]$. Now given $Y = 1$, $Z \sim N(1, \sigma^2)$ and given $Y = -1$ $Z \sim N(-1, \sigma^2)$. Therefore:

   $$p_Z(z) = p.\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-1)^2}{2\sigma^2}} + (1-p).\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z+1)^2}{2\sigma^2}}$$

   and

   $$
   \begin{aligned}
   C_Z(h) &= pe^{jh - \frac{1}{2}h^2\sigma^2} + (1-p)e^{-jh - \frac{1}{2}h^2\sigma^2} \\
   &= e^{-\frac{h^2}{2}\sigma^2}(p(e^{jh} - e^{-jh}) + e^{-jh}) \\
   &= e^{-\frac{h^2}{2}\sigma^2}(\cos h + j(2p-1)\sin h)
   \end{aligned}
   $$

   (b) W=XY

   Let us compute the characteristic function of W

   $$
   \begin{aligned}
   C_W(h) &= C_X(h)p + (1-p)C_{-X}(h) \\
   &= C_X(h)
   \end{aligned}
   $$

   since $C_X(h) = C_{-X}(h)$ as $X = N(0, \sigma^2)$. This is only true because $E[X] = 0$. Therefore W is Gaussian with the same distribution as X.

(c) Now let $U = W + X$. We know that X and W are individually Gaussian but we do not know if they are jointly Gaussian. It is clear by observation that W and X are dependent. To compute the distribution of U we have:

$$\begin{aligned} F_U(z) = \mathbb{P}(U \leq z) &= \mathbb{P}(X(Y+1) \leq z) \\ &= \mathbb{P}(X \leq \frac{z}{2}|Y=1)p + (1-p) = \mathbb{P}(X \leq \frac{z}{2})p + 1 - p \quad z \geq 0 \\ &= p\mathbb{P}(X \leq \frac{z}{2}) \quad z < 0 \end{aligned}$$

which is clearly not a Gaussian distribution This shows that unless we specify the joint distribution, the sum of two Gaussians need not be Gaussian.

Now: $E[W] = 0$ and $var(W) = \sigma^2$. and

$$cov(WX) = E(WX) = E[X^2Y] = \sigma^2(2p-1)$$

If $p = 0.5$ they would be uncorrelated but not independent because they are not jointly Gaussian. In general, if $p \neq 0.5$ they are correlated and not independent.

5. First note that:

$$Z_t = X\cos(2\pi t + \theta) = \cos 2\pi t X \cos\theta - \sin 2\pi X \sin\theta$$

where for eact t , $\cos 2\pi t$ and $\sin 2\pi t$ are just non-random constants. If one defines $X_1 = X\cos\theta$ and $X_2 = X\sin\theta$ then by a simple Jacobian calculation it can be seen X_1 and X_2 are independent Gaussian random variables. Therefore Z_t is a Gaussian r.v. for every t being a linear combination of independent Gaussian r.v's.

Now the joint distribution of $\{Z_{t_1}, Z_{t_2}\}$ can be computed as follows (by taking the moment generating function) $E[e^{h_1 Z_{t_1} + h_2 z_{t_2}}]$ and noting that:

$$\begin{aligned} Z &= aZ_{t_1} + bZ_{t_2} \\ &= aX\cos(2\pi t_1 + \theta) + bX\cos(2\pi t_2 + \theta) \\ &= (a\cos 2\pi t_1 + b\cos 2\pi t_2)X\cos\theta - (a\sin 2\pi t_1 + b\sin 2\pi t_2)X\sin\theta \end{aligned}$$

showing Z is Gaussian for any a and b from the fact that $X\cos\theta$ and $X\sin\theta$ are independent Gaussians. Therefore since h_1 and h_2 are arbitrary, the r.v. in the exponent is a Gaussian and one readily can obtain the conclusion that they are jointly Gaussian.

Indeed one can show it is a Gaussian stochastic process because all finite combinations will be Gaussian.

6. Let us first compute the characteristic function of a Poisson r.v.

$$\begin{aligned} C(h) = E[e^{jhX}] &= \sum_{n=0}^{\infty} e^{jhn}\mathbb{P}(X = n) \\ &= \sum_{n=0}^{\infty} e^{jhn}\frac{\lambda^n}{n!}e^{-\lambda} \\ &= e^{-\lambda(1-e^{jh})} \end{aligned}$$

Now $Y = \sum_{i=1}^n X_i$ where $X_i \sim Poisson(\lambda_i)$ and the $X_i's$ are independent.

Therefore:

$$C_Y(h) = \prod_{i=1}^n C_{X_i}(h) = \prod_{i=1}^n e^{-\lambda_i(1-e^{jh})} = e^{-\sum_{i=1}^n \lambda_i(1-e^{jh})}$$

where the last expression is just the characteristic function of a Poisson r.v. with parameter $\sum_{i=1}^n \lambda_i$.

7. Advanced problem 1.

Let $Z = X + Y$ and X and Y are independent. We want to show that if Z is Gaussian then X and Y are also Gaussian. Indeed let us see the simple case: X and Y are identically distributed. Suppose $Z \sim N(0, \sigma^2)$. then by definition:

$$CZ(h) = C_X(h)C_Y(h) = (C_X(h))^2$$

Therefore the characteristic function of X is just $e^{-\frac{\sigma^2}{4}h^2}$ showing $X \sim N(0, \frac{\sigma^2}{2})$. Of course they need not be identical and so we need to show it more generally.

We show the result via the following idea of stable distributions.

Lemma

Let X and Y be 0 mean variance 1 i.i.d random variables.

Suppose there exist a,b. and c and a random variable Z such that:

$$aX + bY = cZ$$

where Z has the same distribution of X and Y..

Then Z must be $N(0, 1)$.

Proof:

$E[e^{jhcX}] = C(ch)$ where $C(h) = E[e^{jhX}]$. Now from independence of X and Y we have $c^2 = a^2 + b^2$ and $C_(ch) = C(ah)C(bh)$. Define $\phi(x) = \log C(x)$.

Then:

$$\phi(ch) = \phi(ah) + \phi(bh) = \phi(ah + bh)$$

Noting that $c^2 = a^2 + b^2$ we have

$$\phi(x + y) = \phi(x) + \phi(y) = \phi(\sqrt{x + y})$$

The only solution of this equation is of the form:

$$C(x) = Ke^{dx^2}$$

Noting $C(0) = 1$ and $C''(0) = -1$ we obtain $K = 1$ and $d = \frac{1}{2}$ showing that $C(h)$ is the characteristic function of a standard normal random variable.

This thus allows us to conclude that if Z is Gaussian and X and Y are independent then X and Y must be Gaussian by this stability result where $c^2 = var(Z)$ and $var(X) = a^2$, $var(Y) = b^2$ with $c^2 = a^2 + B^2$. or $Z = aX + bY$ where X and Y are independent $N(0, 1)$ random variables.

8. Advanced problem 2

To show that if $f(.)$ is a "nice" function then

$$E[f'(X) - Xf(X)] = 0 \leftrightarrow X \sim N(0,1)$$

Let us show the sufficiency part. Let X be standard normal. Let $E_N[.]$ denote expectation w.r.t. the standard normal.Then

$$E_N[f'(X) - Xf(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f'(x) - xf(x))e^{-\frac{x^2}{2}} dx$$

Integrate by parts the first term on the rhs above.

$$\int_{-\infty}^{\infty} f'(x)e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} xf(x)e^{-\frac{x^2}{2}} dx$$

hence the result follows.

Without loss of generality let us assume that X has a density $p(x)$. Then we obtain:

$$\int_{-\infty}^{\infty} (f'(x) - xf(x))p(x) = 0 = \int_{-\infty}^{\infty} f(x)(p'(x) + xp(x))dx$$

Since $f(.)$ is arbitrary it implies that $p'(x) + xp(x) = 0$ or

$$p(x) = ce^{-\frac{x^2}{2}}$$

From the normalization condition we obtain $c = \frac{1}{\sqrt{2\pi}}$. or $p(x)$ is the density of a $N(0,1)$ r.v.

To show that there exists a function $f(x)$ such that:

$$h(x) - E_N[h(X)] = f'(x) - xf(x)$$

one can readily see that if we define:

$$f(x) = e^{\frac{x^2}{2}} \int_{-\infty}^{x} (h(y) - E_N[h])e^{-\frac{y^2}{2}} dy$$

then $f(.)$ will satisfy:

$$E_N[f'(X) - Xf(X)] = 0$$