ECE 604- PSET 4 Solution

Answers to selected problems only given in detail. Routine problems are not worked out in detail

1. This problem basically shows that any Gauss-Markov process can be viewed as a time-changed Brownian motion.

   a)To show that $X_t = \sqrt{\lambda} W_{\frac{t}{\lambda}}$ is BM it is enough to show $\mathbf{E}[X_t X_s] = \min(t,s)$ which follows easily since $\frac{t}{\lambda}$ is monotone in t.

   b) Suppose $\frac{g(t)}{f(t)}$ is increasing in t then, for $s > t$,

$$\mathbf{E}[X_t X_s] = f(t)f(s)\min\{\frac{g(t)}{f(t)}, \frac{g(s)}{f(s)}\} = f(s)g(t) = e^{-\lambda(s-t)}$$

   Hence the result follows by taking $g(t) = e^{\lambda t}$ and $f(t) = e^{-\lambda t}$.

2. To solve this problem we need to use the Borel-Cantelli lemma (1st part).

   Define $A_n = \{X_n > n\varepsilon\} = \{\frac{X_n}{\varepsilon} > n\}$.

   For any non-negative r.v. X we have:

$$\sum_{m=0}^{\infty} \mathbb{P}(X > m+1) \leq \mathbf{E}[X] = \sum_{m=0}^{\infty} \int_m^{m+1} \mathbb{P}(X > x)dx = \leq \sum_{m=0}^{\infty} \mathbb{P}(X > m)$$

   .

   Therefore we have (using the fact that the $\{X_n\}$ are identically distributed (as say X_1):

$$\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(\frac{X_1}{\varepsilon} > n) \leq \frac{\mathbf{E}[X_1]}{\varepsilon} < \infty$$

   Hence by Borel-Cantelli lemma, $\{A_n\}$ $i.o.$ is 0 a.s. or $\frac{X_n}{n} > \varepsilon$ only a finite number of times and hence goes to 0 almost surely.

3. This is a very useful result for showing mean square convergence.

   Suppose $\lim_{n,m\to\infty} \mathbf{E}[X_n X_m] = C$ then we have:

$$\begin{aligned}\lim_{n,m\to\infty} \mathbf{E}[X_n - X_m]^2 &= \lim_{n,m\to\infty} \left\{\mathbf{E}[X_n^2] - 2\mathbf{E}[X_n N_m] + \mathbf{E}[X_m^2]\right\} \\ &= C - 2C + C = 0\end{aligned}$$

   On the other hand if $\lim_{n,m\to\infty} \mathbf{E}[X_n - X_m]^2 \to 0$ then it necessarily follows that

$$\lim_{n,m\to\infty} \mathbf{E}[X_n X_m] = \lim_{n\to\infty} \mathbf{E}[X_n^2] = \lim_{m\to\infty} \mathbf{E}[X_m^2] = C$$

   Consider the Gauss-Markov process:

$$X_{n+1} = aX_n + bw_n$$

with $|a| > 1$. Then we know $R_{n+1} = a^2 R_n + b^2 \to \infty$ and so $\{X_n\}$ does not converge in the mean square.

To show that $\frac{X_n}{a^n} = Y_n$ converges in the mean square we need to show that:

$$\lim_n \sup_m \mathbf{E}[(Y_{n+m} - Y_n)^2] \to 0$$

From a computational standpoint it becomes quite messy and so we use the criterion as stated in the first part of the problem.

First note that for $n > m$:

$$X_n = a^{n-m} X_m + \sum_{k=m+1}^{n} a^{n-k} b W_{k-1}$$

Therefore:

$$\mathbf{E}[Y_n Y_m] = a^{-2m} \mathbf{E}[Xm^2] = \frac{1}{a^{2m}}(a^{2m} R_0 + \sum_{k=1}^{m} (a^2)^{k-1} b^2) = R_0 + \frac{b^2}{a^2 - 1} + o(a^{-2m})$$

Therefore as $m \to \infty$ the r.h.s $\to$ $R_0 + \frac{b^2}{a^2-1} = C$ and hence the result follows.

4. This problem directly follows from the Strong Law of Large Numbers (SLLN).

   Indeed let I_i denote the $i - th$ sub-interval. Then:

$$Z_m(i) = \sum_{k=1}^{m} \mathbf{1}_{[X_k \in I_i]}$$

   Then: by the SLLN $\frac{Z_m}{m} \to p_i$ a.s.

   Now

$$\log R_m = \sum_{i=1}^{n} Zm(i) \log p_i$$

   and hence:

$$\frac{\log R_m}{m} = \sum_{i=1}^{n} \frac{\sum_{k=1}^{m} \mathbf{1}_{[X_k \in I_i]}}{m} \log p_i \to \sum_{i=1}^{n} p_i \log p_i = -h \quad as \quad m \to \infty$$

5. This problem is just one where we compute the Fourier transform and see whether it is non-negative for all $\omega \in (-\infty, \infty)$. Because the functions are only functions of t-s we need to consider the functions $R(t)$.

   The answers to these are therefore: a) Yes. b) Yes c) No.

6. The answer is from the hint. It is a covariance function truncated at some $t < T$ and by computing its Fourier transform we see it can be negative and hence the truncated function cannot be a covariance function.

7. Here the $R(n)$ are the Fourier coefficients (Helmholtz theorem) and we can see that the corresponding spectral density is given by:

$$S(\omega) = R(0) + 2R(5)\cos 5\omega + 2R(15)\cos 15\omega$$

   where $\omega = 2\pi\lambda$. Hence we see that for $\omega = \pi$ we have: $S(\omega) = \pi - 4 - 6 < 0$ so $S(\omega)$ is negative and hence cannot be a spectral density.

8. This is easy. Define $Y_t = \sqrt{a} X_{at}$ where $cov(X_t, X_s) = R(t,s)$. Then $cov(Y_t, Y_s) = aR(at, as)$ and thus $aR(at, as)$ is a bona fide covariance.

9. First note that:
$$\mathbf{E}[e^{j\omega W_t}] = e^{-\frac{1}{2}\omega^2 t}$$
.

Now
$$\begin{aligned}
\mathbf{E}[X(t)] &= \frac{1}{2j}\mathbf{E}[e^{j(2\pi f t + W_t)} - e^{-j(2\pi f t + W_t)}] \\
&= e^{-\frac{t}{2}} \sin(2\pi f t) \\
&\rightarrow 0 \quad as \quad t \rightarrow \infty
\end{aligned}$$

Now we need to compute the covariance and show that $\lim_{t\rightarrow\infty} R(t, t+T) = S(T)$ where $S(T)$ is a covariance function.

Now:
$$R(T, T+t) = \mathbf{E}[X(T)X(T+t)] - \mathbf{E}[X(T)]\mathbf{E}[X(T+t)]$$

From the calculations above we see that both $\mathbf{E}[X(T)]$ and $\mathbf{E}[X(t+T)]$ go to 0 as $T \rightarrow infty$ so we only need to consider the first term.

Now:
$$\begin{aligned}
\mathbf{E}[X(T)X(T+t)] &= \mathbf{E}[\sin(2\pi f T + W_T)\sin(2\pi f(T+t) + W_{T+t})] \\
&= \frac{1}{2}\mathbf{E}[\cos(2\pi f t + W_{T+t} - W_T)] - \frac{1}{2}\cos(2\pi f(2T+t) + W_T + W_{t+T})]
\end{aligned}$$

Now as in the first part $\lim_{T\rightarrow\infty} \mathbf{E}[\cos(2\pi f(2\pi(2T+t) + W_T + W_{T+t})] \rightarrow 0$ and $W_{T+t} - W_T$ has the same distribution as W_t hence:
$$\lim_{T\rightarrow\infty} R(T, T+t) = R(t) = \frac{1}{2}\mathbf{E}[\cos(2\pi f t + W_t)] = \frac{1}{2}e^{-\frac{t}{2}}\cos 2\pi f t$$
and so the process is asymptotically w.s.s.

10. $Y(t) = e^{X(t)}$ where $X(t)$ is a w.s.s. Gaussian process with mean m and covariance $R(t)$.

Now:
$$\mathbf{E}[Y(t)] = \mathbf{E}[X(t)] = e^{m + \frac{1}{2}R(0)}$$
where we have used the fact that if X is $N(m, \sigma^2)$, $\mathbf{E}[e^X] = e^{m+\frac{1}{2}\sigma^2}$.

Hence :
$$\begin{aligned}
cov(Y(s)Y(s+t)] &= \mathbf{E}[Y(s)Y(s+t)] = \mathbf{E}[Y(s)]\mathbf{E}[Y(s+t)] \\
&= \mathbf{E}[Y(s)Y(s+t)] - e^{2m+R(0)}
\end{aligned}$$

Now:
$$\begin{aligned}
\mathbf{E}[Y(s)Y(s+t)] &= \mathbf{E}[e^{X(s)+X(s+t)}] \\
&= e^{2m + \frac{1}{2}(2R(0) + 2R(t))} \\
&= e^{2m + R(0) + R(t)}
\end{aligned}$$

Hence

$$cov(Y(s)Y(s+t)) = e^{2m+R(0)}\left(e^{R(t)}-1\right)$$

A sufficient condition for the spectral density to exist is:

$$\int_{-\infty}^{\infty} e^{R(t)}dt < \infty$$

There is no simple answer without further knowledge of $R(t)$.