Lecture 3: Palm Probabilities and Rate Conservation Laws

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Palm probabilities and stationary queueing systems

The idea of Palm probabilities is one of conditioning on a point in time where an *event* takes place.

Let $\{T_n\}$ denote a sequence of r.v.'s such that $...T_{-1} < T_0 \leq 0 < T_1 <, \cdots$. The r.v's correspond to a sequence of time points. Assume that the sequence is stationary i.e $T_{i+1} - T_i$ are identically distributed.

Define:

$$N_t = N(0,t) = \sum_k \mathbb{I}_{[0,t)}(T_k)$$

then N_t is said to be a simple point process (it counts how many points lie in $[0,t)$. Now suppose $\{X_t\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}_t, P)$ on which also $\{N_t\}$ is defined.

A Palm probability tries to make sense of the following:

$$\mathbb{P}^t(X_t \in A | \Delta N_t = 1)$$

i.e. the probability of $X_t \in A$ when a point occurs.

Note the event $\Delta N_t = 1$ occurs on a set of measure 0 and thus making sense of such a conditional probability needs some care.

Let us see some examples.

Suppose $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity λ i.e. $N_t - \lambda t$ isa $\mathcal{F}_t$ martingale. Consider X_t too be $\mathcal{F}_t$ adapted.

$$\begin{aligned}
\mathbb{E}[X_t|\Delta N_t = 1] &= \lim_{\delta\to 0} \frac{\mathbb{E}[X_t \mathbb{1}_{[N[t,t+\delta]=1]}]}{\mathbb{E}[\mathbb{1}_{(N[t,t+\delta)=1]}} \\
&= \lim_{\delta\to 0} \frac{\mathbb{E}[X_t \mathbb{E}[\mathbb{1}_{[N[t,t+\delta)=1]}|\mathcal{F}_t]}{\mathbb{E}[\mathbb{1}_{[N[t,t+\delta=1]})|\mathcal{F}_t]}
\end{aligned}$$

By definition of the stochastic intensity $\mathbb{E}[\mathbb{1}_{[N(t,t+\delta)=1]}|\mathcal{F}_t] = \lambda\delta + o(\delta)$.

And hence we see that $\mathbb{E}[X_t|\Delta N_t = 1] = \mathbb{E}[X_t]$

We can make this argument completely rigorous. The key is that conditioning w.r.t points of a Poisson process do not affect the probabilities. This is an apparition of the so-called PASTA property. We will see this more in detail later.

In general conditioning does affect the expectation.

Let us see another example. Once again let $\{N_t\}$ be a Poisson process with intensity λ.

Let $\{X_t\}$ be a stochastic process adapted to $\mathcal{F}_t$ then

$$\mathbb{E}[\int_0^t X_{s-} dN_s = \mathbb{E}[\sum_n X_{T_n-} \mathbb{1}_{[T_n \leq t]} = \lambda \int_0^t \mathbb{E}[X_s] ds$$

This is sometimes called Campbell's formula. In this case it just follows from the martingale property.

However when N_t is a stationary point process we can still obtain a similar formula if we replace the expectation on the r.h.s by expectation w.r.t. Palm probability and λ by $\mathbb{E}[N[0,1)]$

Let us now see a more general situation in the discrete-time case (when there is no problem in defining the conditional probability). Let $\{\xi_n\}$ be a a stationary sequence of $\{0,1\}$ r.v. with $\mathbb{P}(\xi_n = 1) = \lambda$. Let $\{T_n\}$ be the set of times when $\xi_n = 1$ and we adopt the convention $\cdots < T_{-1} < T_0 \leq 0 < T_1 < \cdots$. Let $N(n) = \xi_n$.

We can now define for any $A \in \mathcal{F}$:

$$\begin{aligned}
\mathbb{P}^n(X_n \in A) &= \mathbb{P}(X_n \in A | N(n) = 1) \\
&= \frac{\mathbb{P}(X_n \in A, N(n) = 1)}{\mathbb{P}(N(n) = 1)} \\
&= \frac{1}{\lambda} \mathbb{P}(X_n \in A, N(n) = 1)
\end{aligned}$$

The probability on the r.h.s is well defined since N_k, X_k are jointly defined.

Note by convention we take $P^0(T_0 = 0) = 1$

Now from above it follows:

$$\begin{aligned}\mathbb{E}[\sum_n X_{T_n} 1\!\!1_{[0 \leq T_n \leq k]}] &= \sum_0^k \mathbb{E}[X_n 1\!\!1_{[N(\{n\}=1]}] \\ &= \lambda \sum_0^k \int dp^n(x) = \lambda(k+1)E^0[X_0]\end{aligned}$$

by the definition of Palm probability above and stationarity.

This is exactly the analog of the result previously.

In the continuous time case we can do the following: Let $\lambda = \mathbb{E}[N[0,1)]$ Now clearly for any r.v. X we can define a measure $\mu(.)$ for $A \in \Re$ as follows:

$$\mu(A) = \frac{1}{\lambda}\mathbb{E}[X \sum_{T_n} \mathbb{I}_{[T_n \in A]}]$$

This is absolutely continuous w.r.t. Lebesgue measure and hence by the Radon-Nikodym theorem we can define a density, say $p^0(t)$. And $\mu(A) = \int_A p^0(s)ds$ where $p^0(t) = \mathbb{E}[X|N(\{t\} = 1]$

Hence in particular:

$$\mathbb{E}[XN(A)] = \lambda \int_A E^t[X]dt$$

This is a special case of the Campbell-Mecke formula.

In lecture 3 we will see these concepts more rigorously.

Inversion Formula and the Waiting Time Paradox

In general how do we relate the Palm probability and the reference probability?

This is given by the inversion formula:

$$\mathbb{E}[X_k] = \mathbb{E}[X_0] = \lambda E^0[\sum_{k=0}^{T_1-1} X_k]$$

Proof: First note by definition of the Palm probability above:

$$\begin{aligned}\lambda \mathbb{E}^0[\sum_{k=0}^{T_1-1} X_k] &= \mathbb{E}[\sum_{k=0}^{T_1-1} X_k \mathbb{I}_{[T_0=0]}] \\ &= \mathbb{E}[\sum_{k=0}^{\infty} X_k \mathbb{I}_{\xi_0=1;\xi_1,\xi_2,..,\xi_{k-1}=0]}] \\ &= \mathbb{E}[X_0 \sum_{k=0}^{\infty} \mathbb{I}_{[\xi_{-k}=1;\xi_{-k+1},\dots\xi_{-1}=0]}]\end{aligned}$$

where we have used stationarity in the last step. Now $\sum_{k=0}^{\infty} \mathbb{I}_{[\xi_{-k}=-1;\xi_m=0,-k+1\leq m\leq 1]} = 1$ a.s. since by definition $\lambda < \infty$ and this just corresponds to stating that there exists a point before 0 at a finite distance. Hence the result follows.

A simple consequence of this result is $\mathbb{E}^0[T_1 - T_0] = \frac{1}{\lambda}$ obtained by taking $X_k = 1$.

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In continuous-time the corresponding result is:

$$\mathbb{E}[X_t] = \lambda \mathbb{E}^0[\int_0^{T_1} X_s ds]$$

where $\lambda = E[N[0,1)]$

Let us now see a consequence of the inversion formula: the famous *inspection paradox*.

Let N_t be a point process.

Define: $A(t) = T_{N_t+1} - t$. Then $A(t)$ is the forward recurrence time- time to the next point given we arrive at t.

Similarly define $B(t) = t - T_{N_t}$ the backward recurrence time. Then $A(t) + B(t) = T_{N_t+1} - T_{N_t}$ is the inter-point time interval. By stationarity $\mathbb{E}^0[T_{N_t+1} - T_{N_t}] = \mathbb{E}^0[T_1 - T_0] = \mathbb{E}^0[T_1]$ since under P^0 we have $T_0 = 0$.

Taking $X_t = T_1 - t$ and $X_t = t - T_0$ and using the inversion formula we have:

$$\mathbb{E}[A(t) + B(t)] = \mathbb{E}[T_1 - T_0] = \lambda \mathbb{E}^0[T_1^2]$$

Noting $\lambda = (\mathbb{E}^0[T_1])^{-1}$ and the fact that $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$ we see that $\mathbb{E}[T_1 - T_0] \geq \mathbb{E}^0[T_1 - T_0]$. The exact difference is $\frac{var^0(T_1)}{E^0[T_1]}$. What this says is that observing an interval between two points biases us- i.e. if we arrive at arbitrary time between two points, then we are more likely to arrive in a long interval.

Motivation

Let $\{X(t)\}$, $t \in R$, be a real valued stochastic process and let N be a point process on R. The time average of $\{X(t)\}$ up to time t is

$$T_t = \frac{1}{t} \int_0^t X(s) ds$$

and the event average of $\{X(t)\}$ up to time t is

$$E_t = \frac{1}{N(0,t]} \int_{(0,t]} X(s) N(ds)$$

The latter integral is interpreted as follows :

$$\int_{(0,t]} X(s) N(ds) \; = \; \sum_{n \geq 1} X(T_n) \mathbf{1}_{[T_n \; \leq \; t]}$$

When the processes are stationary and ergodic, (1.1) corresponds to the mean under the stationary measure while the event average (1.2) converges to the mean under a measure termed the Palm probability.

The natural questions are how does one formally define the Palm probability and how does one compute it? What role does it play in queues?

Example: Jump distributions of Markov chains

Let $\{T_n\}$ be points form a (strictly) increasing sequence of jump times of a Markov chain $X(t)$., with the property $\lim_{n\to\infty} T_n = +\infty$. Let $X_n = X(T_n)$ denote the discrete-time Markov chain viewed at the jump times.

Assume that $\{X(t)\}$ is ergodic, with stationary distribution π, so that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{I}_{[X(s)=i]} ds = \pi(i)$$

In general, the imbedded Markov chain $\{X_n\}$ is not ergodic, and when it is (under the condition that $\sum_{i\in E} \pi(i) q_i < \infty$ where q_i^{-1} is the average sojourn time in state i between two jumps)

$$\lim_{t\to\infty} \frac{1}{N((0,t])} \int_0^t \mathbb{I}_{[X(s)=i]} ds = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{I}_{[X_n=i]} = \pi_0(i)$$

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where

$$\pi_0(i) = \frac{\pi(i)q_i}{\sum_{j\in E}\pi(j)q_j} \neq \pi(i)$$

Equality $\pi_0(i) = \pi(i)$ holds for all $i \in E$ if and only if $q_i = \text{constant}$, which is equivalent to $\{T_n\}$ being a Poisson process.

What happens if $x(t)$ is not Markov and the point process is not Poisson?

This will bring us to Palm probabilities.

Palm Probability

Let $(\Omega, F, \mathbb{P})$ be a complete probability space which carries a measurable flow (shift) $\{\theta\}_t$. Let $\mathbb{P}$ be stationary w.r.t. $\{\theta_t\}$ i.e.

$$P \circ {\theta_t}^{-1} = P$$

Let N be a point stationary point process (w.r.t the flow $\{\theta_t\}$ defined on $(\Omega, F, \mathbb{P})$

$$N(\theta_t \omega, C) = N(\omega, C + t)$$

where C is a Borel set in $\Re$.

Let λ_N denote the average intensity of N given by:

$$\lambda_N = E[N(0, 1]]$$

The Palm Probability of (N,P) is defined by:

$$P_N^0(A) = \frac{1}{\lambda_N \ell(C)} E[\int_C \mathbf{1}_A(\theta_s) N(ds))$$

where $\ell(c)$ denotes the Lebesgue measure of C and the definition does not depend on C.

Properties of P_N^0

1. $P_N^0(N\{0\} = 1) = P_N^0[T_0 = 0] = 1$
2. $P_N^0 \circ \theta_{T_n} = P_N^0$
3. $E_N^0[T_1] = \lambda_N^{-1}$.

An immediate consequence of the definition is the so-called Mathes-Mecke formula

$$\lambda_N \mathbb{E}_N^0[\int_{\Re} v(s)ds] = \mathbb{E}[\int_{\Re} v(0) \circ \theta_s N(ds)]$$

for any θ_t compatible process $v(t)$

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Stochastic Intensities and Martingale Representations

Let N be a simple, locally finite point process defined on R_+ and let $\{\mathcal{F}_t\}$ be a history of N satisfying the "usual" conditions. Then there exists a unique nondecreasing process (unique within stochastic equivalence) $\{A(t)\}$, $t \in R_+$, that is $\mathcal{F}_t$-predictable and right continuous such that $A_0 = 0$ and

$$\mathbf{E}[\int_R X(s)N(ds)] = \mathbf{E}[\int_R X(s)A(ds)]$$

for all non-negative $\mathcal{F}_t$ predictable processes $\{X(t)\}$, $t \in R$. The process $\{A(t)\}$ is called the $(P, \mathcal{F}_t)$ compensator of the point process N. Moreover $\Delta A(t) \leq 1$ and $\{\omega \mid lim\ N(t,\omega) = \infty\} = \{\omega \mid lim\ A(t,\omega) = \infty\}$. In fact, the process $\{A(t)\}$ is such that $M = N - A$ is a local martingale .

If A_t is absolutely conttinuous w.r.t Lebesgue measure, its density denoted by λ_t given by $A_t = \int_0^t \lambda_s ds$ is called the F_t-(stochastic) intensity.

If $\{X(t)\}$ is any $\mathcal{F}_t$ predictable process such that for all t,

$$\int_{(0,t]} |X(s)|A(ds) < \infty, \quad \text{a.s.}$$

the process $\{\hat{M}(t)\}$ defined by

$$\hat{M}(t) \triangleq \int_{(0,t]} X(s)\big(N(ds) - A(ds)\big)$$

is a $\mathcal{F}_t$-local martingale.

It is also true that the local martingale $M = N - A$ is locally square integrable and more generally if for all t

$$\int_{(0,t]} |X(s)|^2(1 - \triangle A(s))A(ds) < \infty, \quad \text{a.s.}$$

then $\hat{M}(t)$ is a local square integrable martingale. The condition above condition is automatically satisfied if the process $\{X(t)\}$ is bounded and N is locally finite.

The quadratic variation process of $\hat{M}$ denoted $< \hat{M} >$ is defined as the unique predictable, non-negative and increasing process that makes $\hat{M}^2 - < \hat{M} >$ a local martingale. For the local martingale $\hat{M}$ as defined earlier one has the explicit characterization for the quadratic variation process

$$< \hat{M} >_t = \int_{(0,t]} |X(s)|^2 (1 - \triangle A(s)) A(ds)$$

Martingale SLLN

A. If M_t is a local square integrable martingale with a quadratic variation process $< M >$ and if $< M >_\infty (\omega) < \infty$ then $M_t(\omega)$

B. If M_t is a local square integrable martingale with a quadratic variation process $< M >$ and if $< M >_\infty (\omega) = \infty$ then $\frac{M_t(\omega)}{<M>_t(\omega)} \to 0$

Papangelou's Theorem

One of the fundamental theorems that links the Palm probabilities with the stochastic intensity theory is the Papangelou Theorem.

Theorem : $P_N^0 << P$ on $\mathcal{F}_{0-}$ iff N admits a $\mathcal{F}_t$-intensity $\{\lambda(t)\}$. Moreover, in that case $\lambda(t,\omega) = \lambda(0, \theta_t\omega)$ where

$$\lambda(0) = \lambda_N \frac{dP_N^0}{dP} \Big|_{\mathcal{F}_{0-}}$$

Remark In particular Papangelou's formula can be written as:

$$\lambda_N E_N^0[X] = E[\lambda(0)X]$$

for all non-negative $\mathcal{F}_{0-}$ measurable r.v. X.

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Rate Conservation Formula for càdlàg processes

Via Papangelou's theorem we can define a so-called Rate Conservation Law (RCL) for càdlàg processes that allows us to derive most of the important formulae associated with Palm theory.

To do so let us first obtain a representation for all cadlag processes of bounded variation.

Every càdlàg process $\{X_t\}$ càdlàg having jump discontinuities can be written as:

$$X_t = X_0 + X_t^c + X_t^d$$

where X_t^c is purely continuous (w.r.t. t) and X_t^d is purely discontinuous.

If the number of jumps in each compact interval is finite then the process $\{N_t\}$ defined by:

$$N_t = \sum_{s \leq t} \mathbf{1}_{[X_s \neq X_{s-}]}$$

is a locally finite, simple point process.

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Representation (contd)

Let $\Delta X_t = X_t - X_{t-}$ denote the jump of $\{X_t\}$. Then:

$$X_t^d = \sum_{s \le t} \Delta X_s$$

and by the definition of $\{N_t\}$

$$X_t^d = \int_0^t Y_s dN_s$$

where

$$Y_t = \sum_{n=0}^{\infty} (\Delta X_{T_n}) \mathbf{1}_{[T_n, T_{n+1})}(t)$$

Finally if $\{X_t\}$ is of bounded variation then

$$X_t^c = \int_0^t X_s^+ ds$$

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where X_s^+ denotes the right derivative of $\{X_t\}$.

Therefore we obtain the following evolution equation:

$$X_t = X_0 + \int_0^t X_s^+ ds + \int_0^t Y_s dN_s$$

Rate Conservation Law (RCL)

We now state the main result.

Theorem Let $\{X_t\}$ be a cadlag process of bounded variation that is stationary w.r.t θ_t on $\big(\Omega, \mathcal{F}, \mathrm{P}\big)$. Then:

$$\mathbb{E}[X_0^+] + \lambda_N \mathbb{E}_N^0[\Delta X_0] = 0$$

In particular for any $f(.)$ that is C^1 we have:

$$\mathbb{E}[f'(X_0)X_0^+] + \lambda_N \mathbb{E}_N^0[\Delta f(X_0)] = 0$$

where $\Delta f(X_0) = f(X_0) - f(X_{0-})$.

Applications

The first simple application is the level crossing formula due to Brill and Posner.

Theorem Let $\{X - t\}$ be a stationary cadlag process that possesses a density. Then:

$$p(x)\mathbb{E}[X_0^+/X_0 = x] = \lambda_N \mathbb{E}_N^0[\mathbf{1}_{[X_{0-}>x]} - \mathbf{1}_{[X_0>x]}]$$

Noting that:

$$\mathbf{1}_{[X_{0-}>x]} - \mathbf{1}_{[X_0>x]} = \mathbf{1}_{[X_{0-}>x]}\mathbf{1}_{[X_0\leq x]} - \mathbf{1}_{[X_{0-}\leq x]}\mathbf{1}_{[X_0>x]}$$

We can re-write the result as:

$$p(x)E[X_0^+/X_{0-} = x] = \lambda_N E_N^0[\mathbf{1}_{[X_{0-}>x]}\mathbf{1}_{[X_0\leq x]} - \mathbf{1}_{[X_{0-}\leq x]}\mathbf{1}_{[X_0>x]}]$$

Formulae for Palm probabilities

Now we see two very fundamental formulae arising in Palm theory namely: 1) The Palm inversion formula and 2) Neveu's cycle formula.

Palm Inversion Formula

For any X_t that is compatible with θ_t (stationary):

$$E[X_0] = \lambda_N \mathbb{E}_N^0 [\int_0^{T_1} X_s ds]$$

Proof: Let $T_+(t)$ be the first point of N_t after t.

Define $Y_t = \int_t^{T_+(t)} X_s ds$ Then $Y_t^+ = -X_t$ and $Y_{0-} = 0$ since $T_+(0-) = T_0$ and $T_0 = 0$ under $\mathbb{P}_0^N$ Hence

$$\mathbb{E}[Y_0^+] = -\mathbb{E}[X_0] = -\lambda_N \mathbb{E}_N^0 [\Delta Y_0] = -\lambda_N \mathbb{E}_N^0 [\int_0^{T_1} X_s ds]$$

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Neveu's Exchange Formula

Let N and N' be two stationary point processes defined on $(\Omega, \mathcal{F}, \mathrm{P})$ compatible with θ_t.

Then:

$$\lambda_N \mathbb{E}_0^N[f(0)] = \lambda_{N'} \mathbb{E}_{N'}^0[\int_0^{T_1'} (f \circ \theta_t) dN_t]$$

Proof; We give a proof with a stochastic intensity. Define $g(t) = \int_t^{T_+'(t)} f_s \lambda_s ds$

Then it is easy to see $g^+ t = -f_t \lambda_t$ and $\Delta g(0) = \int_0^{T_1'} f_s dN_s$. Then applying RCL w.r.t. N' and using Papangelou's formula we have:

$$\lambda_N \mathbb{E}_N^0[f(0)] = \lambda_{N'} \mathbb{E}_{N'}[\int_0^{T_1'} f_s dN_s]$$

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Taking $f = 1$ we see $\lambda_{N'} \mathbb{E}^0_{N'}[N[0, T_1']] = \lambda_N$ we have

$$\mathbb{E}_N[f(0)] = \frac{\mathbb{E}^0_{N'}[\int_0^{T_1'} (f \circ \theta_t) dN_t]}{\lambda_{N'} \mathbb{E}^0_{N'}[N[0, T_1']]}$$

which gives the cycle representation (the Palm distribution can be obtained as an average over a selection of points of the original point process).

Actually N and N' do not have to be subsets but only jointly defined and compatible w.r.t θ_t on the same space.

Little's Formula

Consider a queueing system in which arrivals take place as a stationary point process and each arrival at time T_n brings an amount of work σ_n that is a stationary sequence. Assume the arrivals are serviced in the order they arrive (FIFO)

Let W_t denote the workload in the queue at time t

$$W_t = W_0 + \sum_{n=0}^{N_t} \sigma_n - \int_0^t \mathbb{1}(W_s > 0)ds$$

When there exists a stationary distribution (i.e. when $\lambda_N E_N^0[\sigma] < 1$ we obtain the so-called Little's formula given by:

$$\mathbb{E}[Q] = \lambda_N E_N^0[W_0]$$

where Q_t is the number of customers in the queue at time t.

The proof follows by applying the RCL to the total sojourn time process in the queue defined by:

$$V_t = \int_0^t (W_s - (t-s))^+ dN_s$$

Pollaczek-Khinchine Formula

Consider the function $f(W_t) = W_t^2$ and σ_n to be i.i.d. Then applying the RCL to this function we obtain:

$$\mathbb{E}[W_o] = \frac{\lambda_N E_N^0[\sigma_0^2]}{2(1-\rho)}$$

where $\rho = \lambda_N E_N^0[\sigma_0] < 1$

PASTA

PASTA says that if N is poisson then $\mathbb{E}[X_0] = \mathbb{E}^0_n[X_0]$

This readily follows from Papangelou's theorem because if N is Poisson $\lambda_t(\omega) = \lambda = \lambda_N$.

On the other hand via the martingale SLLN for a PASTA type result to hold we do not even require stationarity.

We state the result below:

Theorem: If N is a simple point process with $\mathcal{F}_t$ intensity λ_t and $\{X_t\}$ is a $\mathcal{F}_t - \textit{predictable}$ process then on the set

$$\tilde{\Omega} = \{\omega \mid \lim_{t\to\infty} A(t,\omega) = \infty\}$$

one has the pointwise result

$$\lim_{t\to\infty}\left\{\frac{1}{N(t)}\int_0^t X(s)N(ds) - \frac{1}{A(t)}\int_0^t X(s)A(ds)\right\} = 0$$

In other words $\lim_{t\to\infty} T_t - E_t = 0$ $a.s.$.

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Palm theory for general stationary increasing measures

Because of the high speed of modern networks we actually need to extend Palm theory to general stationary increasing processes.

Let $A(0,t)$ be a continuous increasing process with stationary increments. Let θ_t be a measurable flow and we assume $A_t = A(0,t)$ is compatible with it. We can define a (fluid) Palm measure readily as follows:

For any Borel set C

$$P_A^0(C) = \frac{1}{\lambda_A}\mathbb{E}[\int_{[0,1]} 1\!\!1_C(\theta_s)A(ds)$$

and in particular:

For all $\mathcal{F}$-measurable processes $(Z(t), t \geq 0)$,

$$E\int_{\Re} Z(s) \circ \theta_s A(ds) = \lambda_A E_A \int_{\Re} Z(s)ds.$$

Fluid queues

For $c > 0$, define $(Q(t), t \geq 0)$ as:

$$Q(t) = Q(0) + A(t) - ct + Z(t),$$

where $(Z(t), t \geq 0)$ is an increasing process, null at 0, which satisfies

- For all $t \geq 0$, $Q(0) + A(t) - ct + Z(t) \geq 0$,
- The support of $Z(dt)$ is included in the set $\{s \geq 0, Q(s) = 0\}$.

Define $\rho_A = \frac{\lambda_A}{c}$. Then, under the condition $\rho_A < 1$, it can be shown that there exists a stationary regime for Q, i.e. there is a unique $\{\theta_t\}$ consistent solution defined on the same probability space $(\Omega, \mathcal{F}, P)$.

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Assume that A is a continuous, stationary, increasing random measure with $E[A(0,1]] = \lambda_A$ and $\rho_A = \lambda_A c^{-1} < 1$. Then,

1) For all continuous functions φ

$$cE\varphi(Q(0))1_{\{Q(0)>0\}} = \lambda_A E_A \varphi(Q(0))1_{\{Q(0)>0\}}.$$

2) For all Borel sets B of $\Re$ which do not contain the origin 0,

$$P_A[Q(0) \in B] = {\rho_A}^{-1} P[Q(0) \in B]$$

3) At the origin i.e. when $B = \{0\}$,

$$P_A[Q(0) = 0] = \rho_A^{-1} \left(P[Q(0) = 0] - 1\right) + 1$$

Little's Law for Fluid Queues

Under the hypotheses above:

$$E[Q(0)] = \rho_A E_A[Q(0)]$$

Note unlike the classical Little's law that relates the average number to the average waiting time here we just have a kind of Mecke formula.

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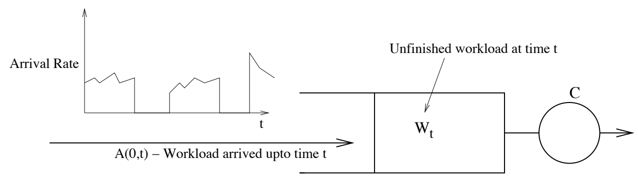
Now consider an input of the type ON-OFF given by the following description:



Let N be a stationary marked point process with points $\{T_n; n \in \mathbf{Z}\}$ and marks $\{(L_n, S_n, F_n), n \in \mathbf{Z}\}$ such that

- $T_0 \leq 0 < T_1$,
- The random marks (L_n) are positive,
- Each triplet (T_n, L_n, S_n) satisfies $T_{n+1} - T_n = L_n + S_n$,
- The marks F_n are continuous increasing processes null at 0, constant on $]L_n, +\infty[$ and such that $F_n(t) \geq ct$ on $\{0 \leq t \leq L_n\}$ (burstiness assumption).

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Define the following random measure:

$$A(B) \equiv \sum_{n \in \mathbf{Z}} \int_{B-T_n} F_n(dt)$$

where B is a Borel set in $\Re$

Then $A(t)$ is a continuous stationary increasing process that specifies the cumulative input up to time t..

We assume that, under P_N, the Palm measure associated with N, the sequences (F_n), (L_n) and (S_n) are i.i.d. and mutually independent and in addition the r.v's S_n are exponentially distributed. Defining $m \equiv E_N[T_1]$,
$n \equiv E_N[F_0(L_0)] = E_N[A(0, T_1]] = \lambda_A m$ and
$q = P[T_0 + L_0 < 0] = \frac{E_N[S_0]}{m}$.

With respect to the filtration $(\mathcal{F}_t)$ generated by the process $A([u,t])_{t\geq 0}, u < t$, the stochastic intensity of the point process (N_t) is given by

$$\lambda_t \equiv (qm)^{-1} 1_{\{\xi_t = 0\}}$$

where $\xi_t \in \{0,1\}$ and takes the value 1 if the source is ON at time t and 0 otherwise.

Pollaczek-Khinchine Formula for Fluid Inputs

$$E[Q(0)] = \frac{1}{c-\lambda_A}\frac{1}{m}E_{N^0}[F_0(L_0)-\lambda_A L_0]^2 - \frac{1}{m}E_{N^0}\left[\int_0^{L_0} t(F_0(dt)-\lambda_A dt)\right]$$

where $m = E_{N^0}[T_1]$, $F_0(t)$ is denotes the cumulative input on $[0,t]$ for the source when ON under P_{N_0}, L_0 is the length of an ON period of the source and $\lambda_A = E[A(0,1)]$.

Note the difference with the Pollaczek-Khinchine formula in the point process case.

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